OPTIMAL DECISION SUPPORT MIXTURE MODEL WITH WEIBULL DEMAND AND DETERIORATION

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ABSTRACT

In this paper, the performance of an inventory model is explored with deteriorating items under an imprecision environment where the demand follows a three-parameter Weibull distribution. Deterioration and holding cost is considered as a linear function of time. Fuzziness has been allowed to deal with imprecision. Mathematical observations of both crisp and fuzzy models have been illustrated to determine the optimal cycle time and optimal inventory cost. The demand distribution, deterioration rate and all costs of models are expressed as triangular, trapezoidal and pentagonal fuzzy numbers. Graded mean integration method is used for defuzzification. Numerical illustrations are provided to validate the applications of the model. Sensitivity analysis with useful graphs and tables are performed to analyze the variability in the optimal solution with respect to change in various system parameters.

KEYWORDS: Weibull Demand, Triangular Fuzzy Number, Trapezoidal Fuzzy Number, Pentagonal Fuzzy Number, Graded Mean Integration

MSC: 90B05

RESUMEN

En este documento, se analiza el rendimiento de un modelo de inventario con elementos deteriorados en un entorno de imprecisión donde la demanda sigue una distribución de Weibull de tres parámetros. El deterioro y el costo de mantenimiento se consideran una función lineal del tiempo. Se ha permitido a la borrosidad lidiar con la imprecisión. Las observaciones matemáticas de los modelos nítidos y difusos se han ilustrado para determinar el tiempo de ciclo óptimo y el costo de inventario óptimo. La distribución de la demanda, la tasa de deterioro y todos los costos de los modelos se expresan como números borrosos triangulares, trapezoidales y pentagonales. Se utiliza el método de integración de medios graduados para la defuzzificación. Se proporcionan ilustraciones numéricas para validar las aplicaciones del modelo. Se realizan análisis de sensibilidad con gráficos y tablas útiles para analizar la variabilidad en la solución óptima con respecto al cambio en varios parámetros del sistema.

1. INTRODUCTION

Most of the existing inventory models based on assumptions that the items can be stored indefinitely to face the future demands. Deteriorating items are common in our daily life. If the rate of deterioration is high, its impact on modeling of such an inventory system cannot be neglected. Deteriorating items refer to the items that become decayed, damaged, evaporative, expired, invalid, devaluation in course of time. But certain types of items either deteriorate or become obsolete with respect to time. The commonly used goods like fruits, vegetables, meat, foodstuffs, fashionable items, alcohol, gasoline, medicines, radioactive substances, photographic films, electronic devices, etc., where deterioration is commonly observed during their normal storage period.

Inventory model with Weibull demand was considered earlier by Tadikamalla [16]. Ghosh et. al. [7] developed an inventory model with Weibull demand rate and production rate is assumed as finite. Tripathy and Pradhan [17] suggested inventory model having Weibull demand and variable deterioration rate. Covert and Philip [3], Giri et. al. [8], Ghosh and Choudhury [6] developed model with Weibull distribution deterioration with various pattern of demand. One of the weaknesses of the current model which is mostly used in business world is the unrealistic assumption of the different parameters. Fuzzy inventory models are more realistic than the traditional inventory models. The uncertainties are due to fuzziness and such cases explained in the fuzzy set theory which was demonstrated by Zadeh [21], Kaufman and Gupta [10]. Syed and Aziz [15] discussed a fuzzy inventory model using signed distance method. Chang et. al. [2], De and Rawat [4] and Jaggi et. al [9]. developed fuzzy models for deteriorating items and demand using triangular fuzzy

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number. Yao and Lee [20] discussed fuzzy inventory with or without backorder for fuzzy order quantity with trapezoidal fuzzy number. Since then many related research were found in Dutta and Kumar [5], Mohanty and Tripathy [11], Nagar and Surana [12], Sahoo et al. [14], Behera and Tripathy [1], Sahoo and Tripathy [13], Tripathy and Sahoo [18], Tripathy and Sukla [19] developed improved inventory model in fuzzy sense using triangular, trapezoidal and pentagonal fuzzy numbers separately. This paper is developed for deteriorating items under imprecision condition where the demand function follows three-parameter Weibull distribution. Deterioration rate and holding cost are considered as time varying linear function. Mathematical models are developed for both crisp and fuzzy sense. The models have been illustrated to determine the optimal cycle time and inventory cost. The demand distribution, deterioration rate and all costs are measured in triangular, trapezoidal and pentagonal fuzzy numbers. The fuzzy model is defuzzified by using graded mean integration method. Empirical investigations are provided to validate the applications of model. Sensitivity analyses have been carried out to study the variability in the optimal solution with respect to change in various system parameters.

2. NOTATIONS AND ASSUMPTIONS

i. \( d(t) \) is the demand rate per unit time.
ii. \( O_c \) is the ordering cost per order.
iii. \( \phi \) is the deterioration rate per unit time.
iv. \( u_c \) is the unit cost per unit time.
v. \( h_c \) is the inventory holding cost per unit per unit time.
vi. \( t_2 \) is the length of the cycle.
vii. \( Q \) is the ordering quantity per unit.
viii. \( s_c \) is the shortage cost per unit time.
ix. \( C(t_1,t_2) \) is the total inventory cost per unit time.
x. \( C^*_{GM}(t_1,t_2) \) is the defuzzified value of \( \tilde{C}(t_1,t_2) \) by applying graded mean integration method using parameters are triangular fuzzy number.
x. \( C^{**}_{GM}(t_1,t_2) \) is the defuzzified value of \( \tilde{C}(t_1,t_2) \) by applying graded mean integration method using parameters are trapezoidal fuzzy number.
xii. \( C^{***}_{GM}(t_1,t_2) \) is the defuzzified value of \( \tilde{C}(t_1,t_2) \) by applying graded mean integration method using parameters are pentagonal fuzzy number.
xiii. Demand \( d(t) = \alpha \beta (t-\gamma)^{\beta-1} \) is three parameter Weibull distribution function where \( \alpha > 0 \) is scale parameter, \( \beta > 0 \) is shape parameter and \( 0 < \gamma < 1 \) is location parameter.
xiv. \( \phi(t) = a + bt \) is the time varying deterioration rate.
xv. \( h_c = p + qt \) is holding cost per unit time.
xvi. The replenishment rate is instantaneous and lead-time is zero.
xvii. Shortages are allowed and fully backlogged.

3. MATHEMATICAL MODEL

Let \( I(t) \) be the on-hand inventory at any time \( t \) with initial inventory \( Q \). During the period \([0, t_1] \) the on-hand inventory is depleted due to demand and deterioration and also completely exhausted at time \( t_1 \). The period \([t_1, t_2] \) is the period of shortages, which are fully backlogged. At any instant of time the inventory level \( I(t) \) is expressed by the differential equation.
3.1. Crisp Model

\[
\frac{dl(t)}{dt} + \phi(t)I(t) = -d(t), \quad 0 \leq t \leq t_1
\]

with \( I(0) = Q \) and \( I(t_1) = 0 \).

And

\[
\frac{dl(t)}{dt} = -d(t), \quad t_1 \leq t \leq t_2
\]

with \( I(t_1) = 0 \)

Solutions of equation (1) and (2) are

\[
I(t) = \left\{ \begin{array}{l}
\alpha \left( (t_1 - \gamma)^\beta (1 - at - \frac{bt}{2} + at_1 + \frac{bt_1}{2}) - \frac{\beta(t_1 - \gamma)^{\beta+1}}{\beta+1} (a + bt_1) \right), \quad 0 \leq t \leq t_1 \\
\alpha((t_1 - \gamma)^\beta - (t - \gamma)^\beta), \quad t_1 \leq t \leq t_2
\end{array} \right.
\]

Inventory is available in the system during the time period \((0, t_1)\). Hence the cost for holding inventory in stock is computed for time period \((0, t_1)\) only.

Total no. of holding cost unit \( IHC \) during period \([0, t_2]\) is given by

\[
IHC = \int_0^{t_1} I(t)dt = p\alpha \left( (t_1 - \gamma)^\beta \left( t_1 + \frac{at_1^2}{2} + \frac{bt_1^2}{3} \right) - \frac{\beta(t_1 - \gamma)^{\beta+1}}{\beta+1} \left( 1 + at_1 + bt_1 \right) + \frac{\beta(t_1 - \gamma)^{\beta+2}}{\beta+2} (a + 2bt_1) \right)
\]

\[
+ q\alpha \left( (t_1 - \gamma)^\beta \left( t_1 + \frac{at_1^2}{2} + \frac{bt_1^2}{3} \right) - \frac{\beta(t_1 - \gamma)^{\beta+1}}{\beta+1} \left( t_1 + \frac{at_1^2}{2} + \frac{bt_1^2}{3} \right) \right) + \frac{3\beta(t_1 - \gamma)^{\beta+4}}{\beta+4} \left( \frac{bt_1}{2} \beta + 3K_2 \right) - a \frac{\beta(t_1 - \gamma)^{\beta+3}}{\beta+3} (a + 3bt_1) \right)
\]

Total no. of deteriorated units \( I_D \) during period \([0, t_2]\) is given by

\[
I_D = u_c[Q - \text{Total Demand}]
\]

\[
= u_c \alpha \left( (t_1 - \gamma)^\beta \left( at_1 + \frac{bt_1^2}{2} \right) - \frac{\beta(t_1 - \gamma)^{\beta+1}}{\beta+1} (a + bt_1) + \frac{\beta(t_1 - \gamma)^{\beta+2}}{\beta+2} \left( \frac{bt_1}{2} \beta + 3K_2 \right) - a \frac{\beta(t_1 - \gamma)^{\beta+3}}{\beta+3} (a + 3bt_1) \right)
\]

Total no. of shortage units \( I_S \) during period \([0, t_2]\) is given by

\[
I_S = s_c \int_{t_1}^{t_2} I(t)dt = -s_c \alpha \left( (t_1 - \gamma)^\beta \left( t_2 - t_1 \right) - \frac{\beta(t_1 - \gamma)^{\beta+1}}{\beta+1} \left( \frac{bt_1}{2} \beta + 3K_2 \right) \right)
\]

Total cost of the system per unit time is given by

\[
C(t_1, t_2) = \frac{1}{t_2} [O_c + IHC + I_D + I_S]
\]

\[
= \frac{1}{t_2} \left[ O_c + p\alpha \left( (t_1 - \gamma)^\beta \left( t_1 + \frac{at_1^2}{2} + \frac{bt_1^2}{3} \right) - \frac{\beta(t_1 - \gamma)^{\beta+1}}{\beta+1} \left( 1 + at_1 + bt_1 \right) + \frac{\beta(t_1 - \gamma)^{\beta+2}}{\beta+2} (a + 2bt_1) \right)
\]

\[
- \frac{2\beta(t_1 - \gamma)^{\beta+3}}{\beta+3} \left( \frac{bt_1}{2} \beta + 3K_2 \right) + \frac{\beta(t_1 - \gamma)^{\beta+4}}{\beta+4} \right)\right]
\]

626
\[
\begin{align*}
&+ \begin{pmatrix}
\frac{3b(t_2 - \gamma)^{\beta_{1+}}}{(\beta+1)(\beta+2)} - \frac{(t_2 - \gamma)^{\beta_{1+}}}{(\beta+1)} (a + b t_1) + \frac{b(t_2 - \gamma)^{\beta_{1+}}}{(\beta+1)} + \frac{a(-\gamma)^{\beta_{1+}}}{(\beta+1)} - \frac{b(-\gamma)^{\beta_{1+}}}{(\beta+1)}
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
&+ s_c \alpha \left( t_1 - \gamma \right)^{\beta} (t_2 - t_1) - \frac{(t_2 - \gamma)^{\beta_{1+}}}{\beta+1} + \frac{(t_1 - \gamma)^{\beta_{1+}}}{\beta+1} 
\end{align*}
\]

(7)

3.2. Fuzzy Model

It is not easy to determine all the system parameters due to uncertainty in the environment. Accordingly it is assumed that some of these parameters namely \(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\alpha}, \tilde{\beta}, \tilde{u}, \tilde{s}, \tilde{p}, \tilde{q}\) may change within some limits are triangular, trapezoidal and pentagonal fuzzy numbers.

Total cost of the system per unit time in fuzzy sense is governed by

\[
\begin{align*}
\tilde{C}(t_1, t_2) &= \frac{1}{t_2} \left[ O_e + \tilde{p} \tilde{\alpha} \left( t_1 - \gamma \right)^{\beta} (t_1 + \frac{a t_1^2}{2} + \frac{b t_1^2}{3}) - \frac{(t_1 - \gamma)^{\beta_{1+}}}{(\beta+1)} (1 + \tilde{a} t_1 + \tilde{b} t_1^2) + \frac{(t_2 - \gamma)^{\beta_{1+}}}{(\beta+1)} (\tilde{a} + 2\tilde{b} t_1) \right]
\end{align*}
\]

\[
\begin{align*}
+ \tilde{q} \tilde{\alpha} \left( t_1 - \gamma \right)^{\beta} (t_1 + \frac{a t_1^2}{2} + \frac{b t_1^2}{3}) - \frac{(t_1 - \gamma)^{\beta_{1+}}}{(\beta+1)} (t_1 + \frac{a t_1^2}{2} + \frac{b t_1^2}{3}) \right)
\end{align*}
\]

\[
\begin{align*}
+ \tilde{q} \tilde{\alpha} \left( t_1 - \gamma \right)^{\beta} (t_1 + \frac{a t_1^2}{2} + \frac{b t_1^2}{3}) - \frac{(t_1 - \gamma)^{\beta_{1+}}}{(\beta+1)} (t_1 + \frac{a t_1^2}{2} + \frac{b t_1^2}{3}) \right)
\end{align*}
\]

\[
\begin{align*}
+ \tilde{q} \tilde{\alpha} \left( t_1 - \gamma \right)^{\beta} (t_1 + \frac{a t_1^2}{2} + \frac{b t_1^2}{3}) - \frac{(t_1 - \gamma)^{\beta_{1+}}}{(\beta+1)} (t_1 + \frac{a t_1^2}{2} + \frac{b t_1^2}{3}) \right)
\end{align*}
\]

\[
\begin{align*}
+ \tilde{q} \tilde{\alpha} \left( t_1 - \gamma \right)^{\beta} (t_1 + \frac{a t_1^2}{2} + \frac{b t_1^2}{3}) - \frac{(t_1 - \gamma)^{\beta_{1+}}}{(\beta+1)} (t_1 + \frac{a t_1^2}{2} + \frac{b t_1^2}{3}) \right)
\end{align*}
\]

\[
\begin{align*}
+ \tilde{q} \tilde{\alpha} \left( t_1 - \gamma \right)^{\beta} (t_1 + \frac{a t_1^2}{2} + \frac{b t_1^2}{3}) - \frac{(t_1 - \gamma)^{\beta_{1+}}}{(\beta+1)} (t_1 + \frac{a t_1^2}{2} + \frac{b t_1^2}{3}) \right)
\end{align*}
\]

(8)

The total fuzzy cost \(\tilde{C}(t_1, t_2)\) is defuzzified by graded mean integration taking system parameters as triangular, trapezoidal and pentagonal fuzzy numbers.

Case-1: (If Parameters are triangular fuzzy numbers)

Considering \(\alpha = (\alpha_1, \alpha_2, \alpha_3)\), \(\beta = (\beta_1, \beta_2, \beta_3)\), \(\gamma = (\gamma_1, \gamma_2, \gamma_3)\), \(a = (a_1, a_2, a_3)\), \(b = (b_1, b_2, b_3)\), \(u_1 = (u_1, u_2, u_3)\), \(s_c = (s_1, s_2, s_3)\), \(p = (p_1, p_2, p_3)\) and \(q = (q_1, q_2, q_3)\) are triangular fuzzy numbers, the total cost is calculated as

\[
\begin{align*}
C^*_{GM}(t_1, t_2) &= \frac{1}{6} \left( C^*_{GM_1}(t_1, t_2) + 4C^*_{GM_2}(t_1, t_2) + C^*_{GM_3}(t_1, t_2) \right)
\end{align*}
\]

where

\[
\begin{align*}
C^*_{GM_1}(t_1, t_2) = \frac{1}{t_1/2} \left[ O_e + p_1 \alpha_1 \left( t_1 - \gamma \right)^{\beta} (t_1 + \frac{a t_1^2}{2} + \frac{b t_1^2}{3}) - \frac{(t_1 - \gamma)^{\beta_{1+}}}{(\beta+1)} (1 + q_1 t_1 + b t_1^2) \right]
\end{align*}
\]

\[
\begin{align*}
+ \frac{(t_1 - \gamma)^{\beta_{1+}}}{(\beta+1)(\beta+2)} (a_1 + 2b t_1) - \frac{2b(t_1 - \gamma)^{\beta_{1+}}}{(\beta+1)(\beta+2)(\beta+3)}
\end{align*}
\]

\[
\begin{align*}
+ \frac{(-\gamma)^{\beta_{1+}}}{\beta+1} - \frac{a(-\gamma)^{\beta_{1+}}}{(\beta+1)(\beta+2)(\beta+3)} + \frac{b(-\gamma)^{\beta_{1+}}}{(\beta+1)(\beta+2)(\beta+3)}
\end{align*}
\]

627
The optimal values of \( t_1 \) and \( t_2 \) can be obtained so as to minimize the total fuzzy cost \( C^*_{GM}(t_1, t_2) \) by solving the following equations

\[
\frac{\partial C^*_{GM}(t_1, t_2)}{\partial t_1} = 0 \quad \text{and} \quad \frac{\partial C^*_{GM}(t_1, t_2)}{\partial t_2} = 0
\]

Further, for the convexity of total fuzzy cost function \( C^*_{GM}(t_1, t_2) \), the following conditions must be satisfied

\[
\frac{\partial^2 C^*_{GM}(t_1, t_2)}{\partial t_1^2} > 0, \quad \frac{\partial^2 C^*_{GM}(t_1, t_2)}{\partial t_2^2} > 0
\]

and

\[
\left( \frac{\partial^2 C^*_{GM}(t_1, t_2)}{\partial t_1^2} \right) \left( \frac{\partial^2 C^*_{GM}(t_1, t_2)}{\partial t_2^2} \right) - \left( \frac{\partial^2 C^*_{GM}(t_1, t_2)}{\partial t_1 \partial t_2} \right)^2 > 0
\]

It is difficult to prove the convexity mathematically, since it is complicated to determine the second derivatives of the total fuzzy cost function \( C^*_{GM}(t_1, t_2) \). Therefore it is constrained to show the convexity of total fuzzy cost in graph (Figure 5).

**Case-2: (If Parameters are trapezoidal fuzzy numbers)**

Assuming \( \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \), \( \beta = (\beta_1, \beta_2, \beta_3, \beta_4) \), \( \gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \), \( a = (a_1, a_2, a_3, a_4) \), \( b = (b_1, b_2, b_3, b_4) \), \( u_2 = (u_1, u_2, u_3, u_4) \), \( s_c = (s_1, s_2, s_3, s_4) \), \( p = (p_1, p_2, p_3, p_4) \) and \( q = (q_1, q_2, q_3, q_4) \) are trapezoidal fuzzy numbers, the total cost is evaluated as

\[
C^*_{GM}(t_1, t_2) = \frac{1}{6} \left( C^*_{GM_1}(t_1, t_2) + 2C^*_{GM_2}(t_1, t_2) + 2C^*_{GM_3}(t_1, t_2) + C^*_{GM_4}(t_1, t_2) \right)
\]

where

\[
C^*_{GM_1}(t_1, t_2) = \frac{1}{t_2} \left[ O_c + p_i \alpha_i + \frac{(t_1 - \gamma_i) b_i^2}{2} (t_1 + a_i t_1 + b_i t_1^2) \right.
\]

\[
\left. + \frac{3b_i (t_1 - \gamma_i) b_i^2}{(\beta_i + 1)(\beta_i + 2)(\beta_i + 3)(\beta_i + 4)} \right]
\]
\( + u_i \alpha_i \left( t_1 - \gamma_1 \right)^{\beta_i} \left( a_i t_1 + \frac{b_i t_1^2}{2} \right) - \frac{(t_1 - \gamma_1)^{\beta_i+1}}{(\beta_i+1)} (a_i + b_i t_1) + \frac{h_i(t_1 - \gamma_1)^{\beta_i+2}}{(\beta_i+1)(\beta_i+2)} + \frac{a_i(-\gamma_1)^{\beta_i+1}}{(\beta_i+1)} - \frac{h_i(-\gamma_1)^{\beta_i+2}}{(\beta_i+1)(\beta_i+2)} \right) \\
- s_i \alpha_i \left( t_1 - \gamma_1 \right)^{\beta_i} (t_2 - t_1) - \frac{(t_2 - \gamma_1)^{\beta_i+1}}{\beta_i+1} + \frac{(t_1 - \gamma_1)^{\beta_i+1}}{\beta_i+1} \right)

\( C_{GM}^{**}(t_1, t_2), C_{GM}^{**}(t_1, t_2) \) and \( C_{GM}^{*}(t_1, t_2) \) can be written as the pattern of equation (15)

The optimal values of \( t_1 \) and \( t_2 \) can be obtained so as to minimize the total fuzzy cost \( C_{GM}^{**}(t_1, t_2) \) by solving the following equations

\[
\frac{\partial C_{GM}^{**}(t_1, t_2)}{\partial t_1} = 0 \quad \text{and} \quad \frac{\partial C_{GM}^{**}(t_1, t_2)}{\partial t_2} = 0
\]

Further, for the convexity of total fuzzy cost function \( C_{GM}^{**}(t_1, t_2) \), the following conditions must be satisfied

\[
\frac{\partial^2 C_{GM}^{**}(t_1, t_2)}{\partial t_1^2} > 0, \quad \frac{\partial^2 C_{GM}^{**}(t_1, t_2)}{\partial t_2^2} > 0
\]

and

\[
\left( \frac{\partial^2 C_{GM}^{**}(t_1, t_2)}{\partial t_1^2} \right) \left( \frac{\partial^2 C_{GM}^{**}(t_1, t_2)}{\partial t_2^2} \right) - \left( \frac{\partial^2 C_{GM}^{**}(t_1, t_2)}{\partial t_1 \partial t_2} \right) > 0
\]

It is difficult to prove the convexity mathematically, since it is complicated to determine the second derivatives of the total fuzzy cost function \( C_{GM}^{**}(t_1, t_2) \). Therefore it is constrained to show the convexity of total fuzzy cost in graph (Figure-6).

**Case-3: (If Parameters are pentagonal fuzzy numbers)**

Assuming \( \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \), \( \beta = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5) \), \( \gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) \),

\( a = (a_1, a_2, a_3, a_4, a_5), b = (b_1, b_2, b_3, b_4, b_5) \), \( u_c = (u_1, u_2, u_3, u_4, u_5) \), \( s_c = (s_1, s_2, s_3, s_4, s_5) \),

\( p = (p_1, p_2, p_3, p_4, p_5) \) and \( q = (q_1, q_2, q_3, q_4, q_5) \) are pentagonal fuzzy numbers, the total cost is evaluated as

\[
C_{GM}^{**}(t_1, t_2) = \frac{1}{12} \left( C_{GM}^{**}(t_1, t_2) + 3C_{GM}^{**}(t_1, t_2) \right) + 4C_{GM}^{**}(t_1, t_2) + 3C_{GM}^{**}(t_1, t_2) + C_{GM}^{**}(t_1, t_2)
\]

where

\[
C_{GM}^{**}(t_1, t_2) = \frac{1}{12} \left[ O_c + p_c \alpha_1 \left( t_1 - \gamma_1 \right)^{\beta_1} \left( t_1 + \frac{a_i t_1^2}{2} + \frac{b_i t_1^2}{8} \right) - \frac{(t_1 - \gamma_1)^{\beta_i+1}}{(\beta_i+1)} \left( t_1 + \frac{a_i t_1^2}{2} + \frac{b_i t_1^2}{8} \right) \right]
\]

\[
+ q_c \alpha_1 \left( t_1 - \gamma_1 \right)^{\beta_1} \left( t_1 + \frac{a_i t_1^2}{2} + \frac{b_i t_1^2}{8} \right) - \frac{(t_1 - \gamma_1)^{\beta_i+1}}{(\beta_i+1)} \left( t_1 + \frac{a_i t_1^2}{2} + \frac{b_i t_1^2}{8} \right) \right)
\]

\[
+ u_c \alpha_1 \left( t_1 - \gamma_1 \right)^{\beta_1} \left( t_1 + \frac{a_i t_1^2}{2} + \frac{b_i t_1^2}{8} \right) - \frac{(t_1 - \gamma_1)^{\beta_i+1}}{(\beta_i+1)} \left( t_1 + \frac{a_i t_1^2}{2} + \frac{b_i t_1^2}{8} \right) \right)
\]

\[
- s_c \alpha_1 \left( t_1 - \gamma_1 \right)^{\beta_1} \left( t_1 + \frac{a_i t_1^2}{2} + \frac{b_i t_1^2}{8} \right) - \frac{(t_1 - \gamma_1)^{\beta_i+1}}{(\beta_i+1)} \left( t_1 + \frac{a_i t_1^2}{2} + \frac{b_i t_1^2}{8} \right) \right)
\]

\[
- \left( t_1 - \gamma_1 \right)^{\beta_1} \left( t_1 + \frac{a_i t_1^2}{2} + \frac{b_i t_1^2}{8} \right) - \frac{(t_1 - \gamma_1)^{\beta_i+1}}{(\beta_i+1)} \left( t_1 + \frac{a_i t_1^2}{2} + \frac{b_i t_1^2}{8} \right) \right)
\]
Case

Fuzzy Model (Graded Mean Integration Method)

Numerical

Crisp Model

Further, for the convexity of total fuzzy cost function \( C^{***GM} (t_1,t_2) \), the following conditions must be satisfied

\[
\frac{\partial^2 C^{***GM} (t_1,t_2)}{\partial t_1^2} > 0 , \frac{\partial^2 C^{***GM} (t_1,t_2)}{\partial t_2^2} > 0
\]

and

\[
\left( \frac{\partial^2 C^{***GM} (t_1,t_2)}{\partial t_1^2} \right) \left( \frac{\partial^2 C^{***GM} (t_1,t_2)}{\partial t_2^2} \right) - \left( \frac{\partial^2 C^{***GM} (t_1,t_2)}{\partial t_1 \partial t_2} \right) > 0
\]

It is difficult to prove the convexity mathematically, since it is complicated to determine the second derivatives of the total fuzzy cost function \( C^{***GM} (t_1,t_2) \). Therefore it is constrained to show the convexity of total fuzzy cost in graph (Figure-7).

4. **EMPIRICAL INVESTIGATIONS**

Crisp Model

**Numerical Illustration-1:** To understand the effect of solution process, assuming \( O_c = \text{Rs}\ 200/\text{year} \), \( \alpha = 100 \), \( \beta = 4 \), \( \gamma = 0.01 \), \( a = 3 \), \( b = 0.5 \), \( p = 5 \), \( q = 0.3 \), \( u_c = \text{Rs}\ 12/\text{unit} \) and \( s_c = \text{Rs}\ 6/\text{unit} \) and using Mathematica-9, optimal cycle time is evaluated as \( t_1 = 0.105568\ \text{year}, t_2 = 0.846922\ \text{year} \) and optimum total cost is calculated as \( C(t_1,t_2) = \text{Rs}\ 294.319/\text{year} \).

Fuzzy Model (Graded Mean Integration Method)

**Case-1** (Using triangular fuzzy number)

Numerical Illustration-2: Based on the computational process, considering \( O_c = \text{Rs}\ 200/\text{year} \), \( \tilde{\alpha} = (80,100,120) \), \( \tilde{\beta} = (2,4,6) \), \( \tilde{\gamma} = (0.007,0.01,0.013) \), \( \tilde{\alpha} = (2,3,4) \), \( \tilde{\beta} = (0.3,0.5,0.7) \), \( \tilde{\alpha} = (3,5,7) \), \( \tilde{\gamma} = (0.1,0.3,0.5) \), \( \tilde{u}_c = (9,12,15) \) & \( \tilde{s}_c = (4,6,8) \) and using Mathematica-9, the optimal cycle time is found as \( t_1 = 0.0070004\ \text{year}, t_2 = 0.851986\ \text{year} \) and optimal fuzzy cost is obtained as \( C^{GM} (t_1,t_2) = \text{Rs}\ 294.921/\text{year} \).

**Case-2** (Using trapezoidal fuzzy number)

Numerical Illustration-3: To illustrate the solution process, setting \( O_c = \text{Rs}\ 200/\text{year} \), \( \tilde{\alpha} = (70,90,110,130) \), \( \tilde{\beta} = (1,3,5,7) \), \( \tilde{\gamma} = (0.007,0.009,0.011,0.013) \), \( \tilde{\alpha} = (1.5,2.5,3.5,4.5) \), \( \tilde{\beta} = (0.2,0.4,0.6,0.8) \), \( \tilde{\alpha} = (2,4,6,8) \), \( \tilde{\gamma} = (0.15,0.25,0.35,0.45) \), \( \tilde{u}_c = (9,11,13,15) \) & \( \tilde{s}_c = (3,5,7,9) \) and using Mathematica-9, the optimum time is determined as \( t_1 = 0.1327\ \text{year} \), \( t_2 = 0.861536\ \text{year} \) and optimum fuzzy cost is evaluated as \( C^{***GM} (t_1,t_2) = \text{Rs}\ 294.17/\text{year} \).
Case-3 (Using pentagonal fuzzy number)
Numerical Illustration-4: To demonstrate the effect of solution process, assuming $O_c = \text{Rs } 200/\text{year}$, $\tilde{\alpha} = (60, 80, 100, 120, 140)$, $\tilde{\beta} = (2.3, 4.5, 6)$, $\tilde{\gamma} = (0.006, 0.008, 0.01, 0.012, 0.014)$, $\tilde{\alpha} = (1.8, 2.4, 3, 3.6, 4.2)$, $\tilde{b} = (0.3, 0.4, 0.5, 0.6, 0.7)$, $\tilde{p} = (3, 4, 5, 6, 7)$, $\tilde{q} = (0.18, 0.24, 0.3, 0.36, 0.42)$, $\tilde{u}_c = (8, 10, 12, 14, 16)$, $\tilde{s}_c = (2, 4, 6, 8, 10)$ and using Mathematica-9, the optimum time is determined as $t_1 = 0.0954874 \text{ year}$, $t_2 = 0.848497 \text{ year}$ and optimum fuzzy cost is found as $C^*_{GM}(t_1,t_2) = \text{Rs } 291.012/\text{year}$.

Taking various parameters as triangular, trapezoidal and pentagonal fuzzy numbers, the performance can be compared from the values of $t_1$, $t_2$ and optimal cost as given in Table-1, Table-2 and Table-3 respectively:

<table>
<thead>
<tr>
<th>Parameters are triangular fuzzy number</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>Optimal Fuzzy cost $C^*_{GM}(t_1,t_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\alpha}, \tilde{\beta}, \tilde{p}, \tilde{q}, \tilde{u}_c, \tilde{s}_c$</td>
<td>0.0070004</td>
<td>0.851986</td>
<td>294.921</td>
</tr>
<tr>
<td>$\tilde{\beta}, \tilde{\gamma}, \tilde{\alpha}, \tilde{\beta}, \tilde{p}, \tilde{q}, \tilde{u}_c, \tilde{s}_c$</td>
<td>0.0070032</td>
<td>0.853139</td>
<td>296.758</td>
</tr>
<tr>
<td>$\tilde{\alpha}, \tilde{\alpha}, \tilde{\beta}, \tilde{p}, \tilde{q}, \tilde{u}_c, \tilde{s}_c$</td>
<td>0.104025</td>
<td>0.847168</td>
<td>294.206</td>
</tr>
<tr>
<td>$\tilde{\alpha}, \tilde{\beta}, \tilde{p}, \tilde{q}, \tilde{u}_c, \tilde{s}_c$</td>
<td>0.103374</td>
<td>0.84692</td>
<td>294.319</td>
</tr>
<tr>
<td>$\tilde{b}, \tilde{p}, \tilde{q}, \tilde{u}_c, \tilde{s}_c$</td>
<td>0.105544</td>
<td>0.846922</td>
<td>294.319</td>
</tr>
<tr>
<td>$\tilde{p}, \tilde{q}, \tilde{u}_c, \tilde{s}_c$</td>
<td>0.105568</td>
<td>0.846922</td>
<td>294.319</td>
</tr>
<tr>
<td>$\tilde{q}, \tilde{u}_c, \tilde{s}_c$</td>
<td>0.105568</td>
<td>0.846922</td>
<td>294.319</td>
</tr>
<tr>
<td>$\tilde{u}_c, \tilde{s}_c$</td>
<td>0.105568</td>
<td>0.846922</td>
<td>294.319</td>
</tr>
<tr>
<td>$\tilde{s}_c$</td>
<td>0.105568</td>
<td>0.846922</td>
<td>294.319</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameters are trapezoidal fuzzy number</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>Optimal Fuzzy cost $C^*_{GM}(t_1,t_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\alpha}, \tilde{\beta}, \tilde{p}, \tilde{q}, \tilde{u}_c, \tilde{s}_c$</td>
<td>0.1327</td>
<td>0.861536</td>
<td>294.17</td>
</tr>
<tr>
<td>$\tilde{\beta}, \tilde{\gamma}, \tilde{\alpha}, \tilde{\beta}, \tilde{p}, \tilde{q}, \tilde{u}_c, \tilde{s}_c$</td>
<td>0.134339</td>
<td>0.86711</td>
<td>298.659</td>
</tr>
<tr>
<td>$\tilde{\gamma}, \tilde{\alpha}, \tilde{\beta}, \tilde{p}, \tilde{q}, \tilde{u}_c, \tilde{s}_c$</td>
<td>0.0104267</td>
<td>0.847349</td>
<td>294.118</td>
</tr>
<tr>
<td>$\tilde{\alpha}, \tilde{\beta}, \tilde{p}, \tilde{q}, \tilde{u}_c, \tilde{s}_c$</td>
<td>0.101553</td>
<td>0.846918</td>
<td>294.32</td>
</tr>
<tr>
<td>$\tilde{b}, \tilde{p}, \tilde{q}, \tilde{u}_c, \tilde{s}_c$</td>
<td>0.105523</td>
<td>0.846922</td>
<td>294.319</td>
</tr>
<tr>
<td>$\tilde{p}, \tilde{q}, \tilde{u}_c, \tilde{s}_c$</td>
<td>0.00999998</td>
<td>0.846893</td>
<td>294.328</td>
</tr>
<tr>
<td>$\tilde{q}, \tilde{u}_c, \tilde{s}_c$</td>
<td>0.00999998</td>
<td>0.846893</td>
<td>294.328</td>
</tr>
<tr>
<td>$\tilde{u}_c, \tilde{s}_c$</td>
<td>0.00999998</td>
<td>0.846893</td>
<td>294.328</td>
</tr>
<tr>
<td>$\tilde{s}_c$</td>
<td>0.00999998</td>
<td>0.846893</td>
<td>294.328</td>
</tr>
</tbody>
</table>
The sensitivity analysis is performed by keeping all but one system parameter fixed at a time and study the change in the identified variable by fluctuating it from 25% to 50%.

Table-4: Sensitivity analysis on parameters $\alpha$ and $\gamma$ as triangular, trapezoidal and pentagonal fuzzy numbers

<table>
<thead>
<tr>
<th>Parameters</th>
<th>% change in parameters</th>
<th>$C^*_{GM}(t_1,t_2)$</th>
<th>$C^{**}_{GM}(t_1,t_2)$</th>
<th>$C^{***}_{GM}(t_1,t_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>+50</td>
<td>320.641</td>
<td>321.198</td>
<td>314.496</td>
</tr>
<tr>
<td></td>
<td>+25</td>
<td>308.431</td>
<td>308.562</td>
<td>303.668</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>278.227</td>
<td>277.169</td>
<td>275.605</td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td>256.475</td>
<td>255.771</td>
<td>255.497</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>+50</td>
<td>293.169</td>
<td>292.506</td>
<td>289.177</td>
</tr>
<tr>
<td></td>
<td>+25</td>
<td>294.042</td>
<td>293.335</td>
<td>290.092</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>295.804</td>
<td>295.01</td>
<td>291.938</td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td>296.693</td>
<td>295.855</td>
<td>292.869</td>
</tr>
</tbody>
</table>

The above table indicates that as the value of $\alpha$ increases, fuzzy costs $C^*_{GM}(t_1,t_2)$, $C^{**}_{GM}(t_1,t_2)$ and $C^{***}_{GM}(t_1,t_2)$ increase regularly. But fuzzy costs $C^*_{GM}(t_1,t_2)$, $C^{**}_{GM}(t_1,t_2)$ and $C^{***}_{GM}(t_1,t_2)$ decrease slightly, while the value of $\gamma$ increases.
Figure-1 (Behavior of parameters $\alpha$ and $\gamma$ )

Table-5: Sensitivity analysis on parameters $a$ and $b$
as triangular, trapezoidal and pentagonal fuzzy numbers

<table>
<thead>
<tr>
<th>Parameters</th>
<th>% change in parameters</th>
<th>$C^*_{GM}(t_1,t_2)$</th>
<th>$C^{**}_{GM}(t_1,t_2)$</th>
<th>$C^{***}_{GM}(t_1,t_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>+50</td>
<td>294.782</td>
<td>294.83</td>
<td>291.034</td>
</tr>
<tr>
<td></td>
<td>+25</td>
<td>294.731</td>
<td>294.552</td>
<td>291.026</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>294.504</td>
<td>293.607</td>
<td>290.981</td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td>294.205</td>
<td>292.687</td>
<td>290.902</td>
</tr>
<tr>
<td>$b$</td>
<td>+50</td>
<td>294.921</td>
<td>294.176</td>
<td>291.012</td>
</tr>
<tr>
<td></td>
<td>+25</td>
<td>294.921</td>
<td>294.173</td>
<td>291.012</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>294.648</td>
<td>294.167</td>
<td>291.012</td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td>294.648</td>
<td>294.164</td>
<td>291.011</td>
</tr>
</tbody>
</table>

The above table shows that fuzzy cost $C^*_{GM}(t_1,t_2)$ decreases slowly and moderately with the increase with respect to the parameters $a$ and $b$ respectively. The fuzzy cost $C^{**}_{GM}(t_1,t_2)$ increases gradually and moderately with the increase with respect to the parameters $a$ and $b$ respectively and the fuzzy cost $C^{***}_{GM}(t_1,t_2)$ decreases moderately with the increase with respect to the parameter $a$, but fuzzy cost $C^{***}_{GM}(t_1,t_2)$ insensitively with parameter $b$. 

Figure-2 (Behavior of parameters $a$ and $b$)
According to above results, while the values of $p$ and $q$ increase, the fuzzy cost $C_{GM}^*(t_1,t_2)$ decreases moderately. The value of $p$ increases, the fuzzy cost $C_{GM}^{**}(t_1,t_2)$ increases slowly. But fuzzy cost $C_{GM}^{**}(t_1,t_2)$ insensitive with respect to the parameter $q$. The fuzzy cost $C_{GM}^{***}(t_1,t_2)$ is insensitive with respect to the parameters $p$ and $q$.
<table>
<thead>
<tr>
<th>$S_c$</th>
<th>+50</th>
<th>319.94</th>
<th>319.822</th>
<th>314.766</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>+25</td>
<td>308.28</td>
<td>307.952</td>
<td>303.636</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>278.093</td>
<td>277.597</td>
<td>275.622</td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td>256.607</td>
<td>256.406</td>
<td>255.52</td>
</tr>
</tbody>
</table>

The above table indicates that the fuzzy cost $C^*_{GM}(t_1,t_2)$ is highly sensitive with respect to the parameters $O_c$ and $s_c$, while fuzzy $C^*_{GM}(t_1,t_2)$ is low sensitive with parameter $u_c$. The fuzzy cost $C^{**}_{GM}(t_1,t_2)$ and $C^{***}_{GM}(t_1,t_2)$ are highly sensitive with respect to the parameters $O_c$ and $s_c$, while fuzzy $C^{**}_{GM}(t_1,t_2)$ and $C^{***}_{GM}(t_1,t_2)$ are moderately sensitive with parameter $u_c$.

**Figure-4** (Behavior of parameters $O_c$, $u_c$, & $s_c$)

**Figure-5.** Convexity graph of cost function $C^*_{GM}(t_1,t_2)$ with $t_1$ and $t_2$
6. CONCLUSION

The paper deals with optimal decision model with three-parameter Weibull demand, linear deterioration rate and time dependent holding cost. The demand rate, deterioration rate and various costs are taken as triangular, trapezoidal and pentagonal fuzzy numbers. Graded mean integration method is introduced for defuzzification of total inventory cost under fuzzy sense. The above model with pentagonal fuzzy number deals with more realistic approach for storage of the deteriorating goods or commodities. Numerical illustrations are provided to validate the applications of model. Sensitivity analysis with useful graphs and tables are performed to analyze the variability in the optimal solution with respect to change in various system parameters. After comparison, it is concluded that if system parameters are pentagonal fuzzy number then graded mean integration method provides optimum inventory cost. This model can be improved by introducing stochastic demand, freezing technology and dispatch policy.

Acknowledgements: The authors are grateful to the editor and anonymous referees for their valuable and constructive comments.

REFERENCES


