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A GENERALIZATION OF THE NEW WEIBULL-PARETO DISTRIBUTION

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ABSTRACT
In this paper, a transmuted new Weibull-Pareto distribution (TNWP) is suggested as a generalization of the new Weibull-Pareto distribution. Some mathematical properties of the transmuted new Weibull-Pareto distribution are derived, such as the moments, failure rate and mean residual life functions, order statistics. Also, the maximum likelihood estimators for the transmuted new Weibull-Pareto distribution parameters are provided and its Renyi entropy is proved.

KEYWORDS: New Weibull Pareto distribution; Transmuted distribution; Order statistics; Reliability, Renyi entropy.

MSC: 60E05

RESUMEN
En este paper, una “transmuted new Weibull-Pareto distribution” (TNWP) es sugerida como una generalización de la nueva distribución Weibull-Pareto. Algunas propiedades matemáticas de la “transmuted new Weibull-Pareto distribution” son derivadas, como los momentos, la tasa de fallos y la media residual de la función de vida, estadísticos de orden. También los parámetros de máxima verosimilitud para la “transmuted new Weibull-Pareto distribution” son desarrollados y la entropía de Renyi es probada.

PALABRAS CALVE: Nueva distribución Weibull Pareto, “Transmuted distribution”; estadísticos de orden; fiabilidad, entropía de Renyi.

1. INTRODUCTION

A random variable *X* is said to have a new Weibull-Pareto distribution with parameters δ, β and θ if its cumulative distribution function is given by

$$g(x;\delta,\theta,\beta)=\frac{\delta\beta}{\theta^{\beta}}x^{\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}},x>0,\ \beta>0,\delta>0,\ \theta>0 \qquad (1)$$

The corresponding pdf is

$$G(x;\delta,\theta,\beta)=1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}},\quad x>0,\beta>0,\delta>0,\theta>0.$$
(2)

The expected value and the variance of the NWP random variable are given by

$$E(X)=\frac{\theta}{\sqrt[\beta]{\delta}}\Gamma\left(\frac{\beta+1}{\beta}\right) \text{ and } \mathrm{Var}(X)=2\frac{\theta}{\sqrt[\beta]{\delta^2}}\Gamma\left(\frac{\beta+2}{\beta}\right)-\left[\frac{\theta}{\sqrt[\beta]{\delta}}\Gamma\left(\frac{\beta+1}{\beta}\right)\right]^2$$, respectively.

The hazard rate function of the NWP random variable is $R(x;\delta,\theta,\beta)=\frac{\delta\beta}{\theta^{\beta}}x^{\beta-1}$. For more about the NWP distribution see Nasiru and Luguterah (2015) .

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Shaw and Buckley (2007) suggested the transmutation map method. A random variable X is said to have transmuted distribution if its cumulative distribution function(cdf) is given by

$$F(x)=(1+\lambda)G(x)-\lambda\left[G(x)\right]^2,\ \ |\lambda|\le 1, \tag{3}$$

where $G(x)$ is the cdf of the base distribution. If $\lambda=0$, we have the base distribution of X.
The transmuted map method is considered by researchers in the literature, see as an example transmuted additive Weibull distribution which is suggested by Elbatal and Aryal (2003). Elgarhy et al. (2016) suggested transmuted generalized Lindley distribution. Khan and King (2015) proposed transmuted modified inverse Rayleigh distribution. Aryal and Tsokos (2009) suggested transmuted extreme value distribution with an application to climate data. Merovci et al. (2014) proposed the transmuted generalized inverse Weibull distribution. Al-Omari et al. (2017) suggested Transmuted Janadran distribution.
The rest of this paper is organized as follows. In Section 2 we introduced the transmuted NWP distribution. The reliability analysis and hazard rate function are presented in Section 3. A summary of the distributions of order statistics is given in Section 4. In Section 5 the moment generation function is derived. In Section 6 the maximum likelihood estimates of the suggested TNWP parameters are demonstrated. The Rényi entropy is given in Section 7. Finally, the conclusions are given in Section 8.

2. TRANSMUTED NEW WEIBULL PARETO DISTRIBUTION

The quadratic transmuted map method defined in Equation (5) is used to generalize new Weibull Pareto distribution namely; transmuted new Weibull-Pareto (TNWP) distribution. The cdf of the TNWP distribution is computed by substituting the cdf defined in Equation (2) in Equation (3) to have

$$G(x)=1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left(1-\lambda+\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right),\ x>0,\ \beta>0, \delta>0,\ \theta>0,$$

(4)

Deriving Equation (6) with respect to x will give the pdf of the TNWP distribution as

$$g(x)=\frac{\delta\beta}{\theta^{\beta}}x^{\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left[1-\lambda+2\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]. \tag{5}$$

3. RELIABILITY ANALYSIS

The TNWP distribution can be applied to real data. The reliability and hazard rate functions are defined as:

$$R(x)=1-G(x) \tag{6}$$

and

$$h(x)=\frac{g(x)}{1-G(x)}. \tag{7}$$

Therefore, the reliability and hazard rate functions of the TNWP distribution are, respectively, defined by the following theorem.

THEOREM 1: The reliability and hazard rate functions of the TNWP distribution random variable are, respectively

$$R_{TNWP}(x)=e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}-\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}+\lambda e^{-2\delta\left(\frac{x}{\theta}\right)^{\beta}}$$

(8)

and

$$H_{TNWP}(x)=\frac{\delta\beta x^{\beta-1}\left(1-\lambda+2\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)}{\theta^{\beta}\left(1-\lambda+\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)}. \tag{9}$$

Proof of this theorem is straightforward by substituting the pdf and cdf defined in Equations (4) and (5) in Equations (6) and (7).

4. DISTRIBUTION OF ORDER STATISTICS

Order statistics are essential in many areas of practice and statistical theory. Let $X_1, X_2, ..., X_n$ be a random sample of size n from the pdf $g(x)$ and cdf $G(x)$ defined in (4) and (5), and $X_{[1]}, X_{[2]}, ..., X_{[n]}$ be its order statistics. The pdf of the i^{th} order statistic is defined as

$$g_{[j]}(x) = \frac{n!}{(j-1)!(n-j)!} g(x)\left[G(x)\right]^{j-1}\left[1-G(x)\right]^{n-j}, \quad i = 1,2,...,n.$$

(10)

Let $X_{[1]} = \min\left\{X_1, X_2, ..., X_n\right\}$ and $X_{[n]} = \max\left\{X_1, X_2, ..., X_n\right\}$ with pdfs, respectively, defined as follow

$$f_{[1]}(x) = nf(x)[1-F(x)]^{n-1} \text{ and } f_{[n]}(x) = nf(x)[F(x)]^{n-1}.$$

Also, for $0 < x < 1 \ \beta > 0, \delta > 0, \theta > 0.$ As a special case of (11), the pdf of the minimum and maximum order statistics, respectively, are

$$f_{[1]}(x) = n\left[\frac{\delta\beta}{\theta^{\beta}} x^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \left(1-\lambda+2\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)\right]\left(e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} - \lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} + \lambda e^{-2\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)^{n-1}$$

(11)

and

$$f_{[n]}(x) = n\frac{\delta\beta}{\theta^{\beta}} x^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \left(1-\lambda+2\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left(1-\lambda+\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)\right]^{n-1}$$

(12)

Moreover, the *pdf* of the j th order statistic is given as

$$g_{[j]}(x) = \frac{n!}{(j-1)!(n-j)!}\left(\frac{\delta\beta}{\theta^{\beta}} x^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \left[1-\lambda+2\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]\right)$$

$$\times\left[1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} + \lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} - \lambda e^{-2\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{j-1} \times \left[e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} - \lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} + \lambda e^{-2\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{n-j} \quad (13)$$

5. THE MOMENTS OF THE TNWP RANDOM VARIABLE

In this section, we will derive the r^{th} moment, the mean and the variance of the TNWP random variable.

THEOREM 2: If *X* has a TNWP distribution, then the r^{th} moment is defined as

$$E(X^r) = \left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^r \Gamma\left(\frac{r}{\beta}+1\right)\left[1-\lambda-2\lambda\left(\frac{1}{2^{\frac{1}{\beta}}}\right)^r\right] \quad (14)$$

PROOF:

Assume that *X* is a random variable follows the TNWP distribution, then the r^{th} moment for this random variable is defined as:

$$E(X^r)=\int_0^\infty x^r\left[\frac{\delta\beta}{\theta^\beta}x^{\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^\beta}-\frac{\lambda\delta\beta}{\theta^\beta}x^{\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^\beta}+\frac{2\lambda\delta\beta}{\theta^\beta}x^{\beta-1}e^{-2\delta\left(\frac{x}{\theta}\right)^\beta}\right]dx$$

$$=\int_0^\infty\frac{\delta\beta}{\theta^\beta}x^{r+\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^\beta}dx-\int_0^\infty\frac{\lambda\delta\beta}{\theta^\beta}x^{r+\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^\beta}dx+\int_0^\infty\frac{2\lambda\delta\beta}{\theta^\beta}x^{r+\beta-1}e^{-2\delta\left(\frac{x}{\theta}\right)^\beta}dx$$

The first term of $\int_0^\infty\frac{\delta\beta}{\theta^\beta}x^{r+\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^\beta}dx$ can be calculated as:

$$u=\delta\left(\frac{x}{\theta}\right)^\beta=\frac{\delta}{\theta^\beta}x^\beta;\ x=\frac{\theta}{\delta^{\frac{1}{\beta}}}u^{\frac{1}{\beta}};\ du=\frac{\delta\beta}{\theta^\beta}x^{\beta-1}dx\text{, then }dx=\frac{du}{\frac{\delta\beta}{\theta^\beta}x^{\beta-1}}.$$

Hence,

$$\int_0^\infty\frac{\delta\beta}{\theta^\beta}x^{r+\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^\beta}dx=\int_0^\infty\frac{\delta\beta}{\theta^\beta}\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}u^{\frac{1}{\beta}}\right)^{r+\beta-1}e^{-u}\left[\frac{\delta\beta}{\theta^\beta}\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}u^{\frac{1}{\beta}}\right)^{\beta-1}\right]^{-1}du$$

$$=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^r\int_0^\infty u^{\frac{r}{\beta}}e^{-u}du=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^r\Gamma\left(\frac{r}{\beta}+1\right)$$

$$\int_0^\infty\frac{\delta\beta}{\theta^\beta}x^{r+\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^\beta}dx=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^r\int_0^\infty u^{\frac{r}{\beta}}e^{-u}du=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^r\Gamma\left(\frac{r}{\beta}+1\right),$$

Similarly, the other two terms can be done as:

$$-\int_0^\infty\frac{\lambda\delta\beta}{\theta^\beta}x^{r+\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^\beta}dx=-\lambda\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^r\int_0^\infty u^{\frac{r}{\beta}}e^{-u}du=-\lambda\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^r\Gamma\left(\frac{r}{\beta}+1\right),$$

and

$$\int_0^\infty\frac{2\lambda\delta\beta}{\theta^\beta}x^{r+\beta-1}e^{-2\delta\left(\frac{x}{\theta}\right)^\beta}dx=2\lambda\left(\frac{\theta}{(2\delta)^{\frac{1}{\beta}}}\right)^r\Gamma\left(\frac{r}{\beta}+1\right).$$

Therefore,

$$E(X^r)=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^r\Gamma\left(\frac{r}{\beta}+1\right)-\lambda\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^r\Gamma\left(\frac{r}{\beta}+1\right)-2\lambda\left(\frac{\theta}{(2\delta)^{\frac{1}{\beta}}}\right)^r\Gamma\left(\frac{r}{\beta}+1\right)$$

$$=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^r\Gamma\left(\frac{r}{\beta}+1\right)\left[1-\lambda-2\lambda\left(\frac{1}{2^{\frac{1}{\beta}}}\right)^r\right].$$

Remark: Let *X* has a TNWPD, then as special cases of the r^{th} moment the first and second moments are defined as

$$E(X)=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)\Gamma\left(\frac{1}{\beta}+1\right)\left[1-\lambda-2\lambda\left(\frac{1}{2^{\frac{1}{\beta}}}\right)\right]\quad\text{and}\quad E(X^2)=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^2\Gamma\left(\frac{2}{\beta}+1\right)\left[1-\lambda-2\lambda\left(\frac{1}{2^{\frac{1}{\beta}}}\right)^2\right].$$

Therefore, the variance of *X* is

$$\sigma_{TNWP}^{2}=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^{2}\left\{\Gamma\left(\frac{2}{\beta}+1\right)\left[1-\lambda-2\lambda\left(\frac{1}{2^{\frac{1}{\beta}}}\right)^{2}\right]-\left[\Gamma\left(\frac{1}{\beta}+1\right)\right]^{2}\left[1-\lambda-2\lambda\left(\frac{1}{2^{\frac{1}{\beta}}}\right)\right]^{2}\right\}.$$

(15)

The third and fourth moments are

$$E\left(X^{3}\right)=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^{3}\Gamma\left(\frac{3}{\beta}+1\right)\left[1-\lambda-2\lambda\left(\frac{1}{2^{\frac{1}{\beta}}}\right)^{3}\right],$$

and

$$E\left(X^{4}\right)=\left(\frac{\theta}{\delta^{\frac{1}{\beta}}}\right)^{4}\Gamma\left(\frac{4}{\beta}+1\right)\left[1-\lambda-2\lambda\left(\frac{1}{2^{\frac{1}{\beta}}}\right)^{4}\right].$$

Therefore, the skewness and kurtosis of a random variable, respectively, are defined as

$$Sk=\frac{E\left(X^{3}\right)-3\mu\sigma^{2}-\mu^{3}}{\sigma^{3}} \text{ and } Ku=\frac{E\left(X^{4}\right)-4\mu E\left(X^{3}\right)+6E\left(X^{2}\right)\sigma^{2}+3E\left(X^{4}\right)}{\sigma^{8}}.$$

$$CV=\frac{\frac{(2\delta)^{\frac{1}{\beta}}}{\theta}\sqrt{\begin{array}{l}-2^{-2/\beta}\left[\delta\left(\frac{1}{\theta}\right)^{\beta}\right]^{-2/\beta}\left[-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right]^{2}\Gamma\left[1+\frac{1}{\beta}\right]^{2}\\+4^{-1/\beta}\left[\delta\left(\frac{1}{\theta}\right)^{\beta}\right]^{-2/\beta}\left[-4^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right]\Gamma\left[\frac{2+\beta}{\beta}\right]\end{array}}}{\left[-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right]\Gamma\left[1+\frac{1}{\beta}\right]},$$

$$Sk=\mathrm{e}^{-0.69/\beta}\left[\delta\theta^{-\beta}\right]^{-\frac{1}{\beta}}\frac{\left\{\begin{array}{l}-\left(-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)^{3}\Gamma\left[1+\frac{1}{\beta}\right]^{3}-3(-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda)\Gamma\left[1+\frac{1}{\beta}\right]\\\left[\begin{array}{l}-1\left(-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)^{2}\Gamma\left[1+\frac{1}{\beta}\right]^{2}+\left(-4^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)\Gamma\left[\frac{2+\beta}{\beta}\right]\\+\left(-8^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)\Gamma\left[\frac{3+\beta}{\beta}\right]\end{array}\right]\end{array}\right\}}{\left\{-1\left[\left(-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)^{2}\Gamma\left[1+\frac{1}{\beta}\right]^{2}+\left(-4^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)\Gamma\left[\frac{2+\beta}{\beta}\right]\right]\right\}^{3/2}}$$

,

$$Ku=\frac{e^{2.8/\beta}\left(\delta(\theta)^{\beta}\right)^{4/\beta}\left\{\begin{array}{l}3\left(-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)^{4}\Gamma\left[1+\frac{1}{\beta}\right]^{4}+\left(-16^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)\Gamma\left[\frac{4+\beta}{\beta}\right]\\+6\left(-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)^{2}\Gamma\left[1+\frac{1}{\beta}\right]^{2}\left\{\begin{array}{l}-\left(-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)^{2}\Gamma\left[1+\frac{1}{\beta}\right]^{2}\\+\left(-4^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)\Gamma\left[\frac{2+\beta}{\beta}\right]\end{array}\right\}\\-4\left(-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)\left(-8^{\frac{1}{\beta}}(-1+\lambda)+\lambda\right)\Gamma\left[1+\frac{1}{\beta}\right]\Gamma\left[\frac{3+\beta}{\beta}\right]\end{array}\right\}}{\left(-(-2^{\frac{1}{\beta}}(-1+\lambda)+\lambda)^{2}\Gamma\left[1+\frac{1}{\beta}\right]^{2}+(-4^{\frac{1}{\beta}}(-1+\lambda)+\lambda)\Gamma\left[\frac{2+\beta}{\beta}\right]\right)^{4}}$$

6. MAXIMUM LIKELIHOOD ESTIMATION

The maximum likelihood estimates (MLEs) of the parameters of the *TNWPD* are derived from complete samples only. Let $X_1, X_2, ..., X_n$ be a random sample of size *n* from the *TNWPD* with parameters σ , β , θ and λ , and a pdf of *TNWPD*, then, the likelihood function is given by
then, the likelihood function is given by

$$Lg(x;\delta,\beta,\theta,\lambda)=\prod_{i=1}^{n}\left(\frac{\delta\beta}{\theta^{\beta}}x_i^{\beta-1}e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}-\frac{\lambda\delta\beta}{\theta^{\beta}}x^{\beta-1}e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}+\frac{2\lambda\delta\beta}{\theta^{\beta}}x_i^{\beta-1}e^{-2\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\right)$$

$$=\prod_{i=1}^{n}\frac{\delta\beta}{\theta^{\beta}}x_i^{\beta-1}e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\left[1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\right] \qquad (16)$$

The log-likelihood function is

$$\ln Lg(x,x,...,x;\delta,\beta,\theta,\lambda)=\ln\left\{\prod_{i=1}^{n}\frac{\delta\beta}{\theta^{\beta}}x_i^{\beta-1}e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\left[1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\right]\right\}$$

$$=\ln\left\{\left(\frac{\delta\beta}{\theta^{\beta}}\right)^{n}\prod_{i=1}^{n}x_i^{\beta-1}e^{-\delta\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}}\prod_{i=1}^{n}\left[1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\right]\right\}$$

$$=\ln\left(\frac{\delta\beta}{\theta^{\beta}}\right)^{n}+\ln\prod_{i=1}^{n}x_i^{\beta-1}+\ln e^{-\delta\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}}+\ln\prod_{i=1}^{n}\left[1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\right]$$

$$=\ln\left(\frac{\delta\beta}{\theta^{\beta}}\right)^{n}+\sum_{i=1}^{n}\ln\left(x_i^{\beta-1}\right)-\delta\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}+\sum_{i=1}^{n}\ln\left[1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\right]$$

$$=nln\delta+nln\beta-n\beta ln\theta+(\beta-1)\sum_{i=1}^{n}x_i-\delta\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}+\sum_{i=1}^{n}ln\left[1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\right]$$

$$\frac{\partial lnLg}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} x_i - \delta \sum_{i=1}^{n} \left(\frac{x_i}{\theta}\right)^{\beta} ln\left(\frac{x_i}{\theta}\right) - \delta \sum_{i=1}^{n} \frac{\left(\frac{x_i}{\theta}\right)^{\beta} ln\left(\frac{x_i}{\theta}\right)}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}$$

Setting this derivative to zero, we get

$$\frac{n}{\beta} = \delta \sum_{i=1}^{n} \left(\frac{x_i}{\theta}\right)^{\beta} ln\left(\frac{x_i}{\theta}\right) - \sum_{i=1}^{n} x_i + \delta \sum_{i=1}^{n} \frac{\left(\frac{x_i}{\theta}\right)^{\beta} ln\left(\frac{x_i}{\theta}\right)}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}$$

$$\frac{\partial lnLg}{\partial \theta} = \frac{n\beta}{\theta} - \delta\beta \sum_{i=0}^{n} \frac{x_i^{\beta}}{\theta^{\beta+1}} - \sum_{i=0}^{n} \frac{2\lambda\beta\delta x_i^{\beta} e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}{\left(1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\right)\theta^{\beta+1}}$$

Equating this derivative to zero, we get

$$n\theta^{\beta} = \delta \sum_{i=0}^{n} x_i^{\beta} + \sum_{i=0}^{n} \frac{2\lambda\delta x_i^{\beta} e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}},$$

$$\frac{\partial lnLg}{\partial \delta} = \frac{n}{\delta} - \sum_{i=1}^{n} \left(\frac{x_i}{\theta}\right)^{\beta} - \sum_{i=1}^{n} \left[\frac{2\lambda\left(\frac{x_i}{\theta}\right)^{\beta} e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}\right].$$

Setting this derivative to zero, we get

$$\frac{n}{\delta} = \sum_{i=1}^{n} \left(\frac{x_i}{\beta}\right)^{\beta} + \sum_{i=1}^{n} \left[\frac{2\lambda\left(\frac{x_i}{\theta}\right)^{\beta} e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}\right]$$

$$\frac{\partial lnLg}{\partial \lambda} = \sum_{i=0}^{n} \frac{2e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}} - 1}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}.$$

Equating to zero, we have

$$\sum_{i=0}^{n} \frac{2e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}} - 1}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}} = 0.$$

Therefore, the maximum likelihood estimates of the distribution parameters, β, θ, δ and λ are the solution of the following system of equations:

$$\begin{cases} \frac{n}{\beta} = \delta \sum_{i=1}^{n} \left(\frac{x_i}{\theta}\right)^{\beta} ln\left(\frac{x_i}{\theta}\right) - \sum_{i=1}^{n} x_i + \delta \sum_{i=1}^{n} \frac{\left(\frac{x_i}{\theta}\right)^{\beta} ln\left(\frac{x_i}{\theta}\right)}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}} \\ n\theta^{\beta} = \delta \sum_{i=0}^{n} x_i^{\beta} + \sum_{i=0}^{n} \frac{2\lambda\delta x_i^{\beta} e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}} \\ \frac{n}{\delta} = \sum_{i=1}^{n} \left(\frac{x_i}{\beta}\right)^{\beta} + \sum_{i=1}^{n} \left[\frac{2\lambda\left(\frac{x_i}{\theta}\right)^{\beta} e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}\right] \\ \sum_{i=0}^{n} \frac{2e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}} - 1}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}} \end{cases}$$

which does not have a close form, hence; it must be solved numerically.

7. THE RÉNYI ENTROPY

The entropy of a random variable X with density (x) is a measure of variation of the uncertainty. A large entropy value indicates greater uncertainty in the data. Rényi entropy is defined as

$$E_R(\rho) = \frac{1}{1-\rho} \log \int_0^{\infty} \left(f(x) \right)^{\rho} dx \tag{17}$$

where $\rho > 0$ and $\rho \neq 1$. The Rényi entropy for the TNWP variable X with probability density (x) is given by the following theorem.

THEOREM 4: If a random variable X has the *TNWPD* distribution, then the Rényi entropy of X, is given by

$$E_R(\rho) = \frac{1}{1-\rho}\left[\left[\left\{\begin{array}{l}\log\left(\frac{\delta\beta(\lambda-1)}{\theta^\beta}\right)^\rho + \log\Gamma\left(\frac{\rho(\beta-1)+1}{\beta}\right) \\ +\log\sum_{i=0}^{\rho}\binom{\rho}{i}\left(\frac{2\lambda}{\lambda-1}\right)^{\rho-i}(-1)^i\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\frac{1}{\beta}\left(\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\right)^{\rho(\beta-1)}\end{array}\right\}\right]\right].$$

(18)

PROOF:

$$E_R(\rho) = \frac{1}{1-\rho}\log\int_0^\infty [f(x)]^\rho\, dx$$

$$= \frac{1}{1-\rho}\log\int_0^\infty\left[\frac{\delta\beta(\lambda-1)}{\theta^\beta}x^{\beta-1}e^{-2\delta\left(\frac{x}{\theta}\right)^\beta}\left[\frac{2\lambda}{\lambda-1}-e^{\delta\left(\frac{x}{\theta}\right)^\beta}\right]\right]^\rho dx$$

$$= \frac{1}{1-\rho}\log\left\{\left(\frac{\delta\beta(\lambda-1)}{\theta^\beta}\right)^\rho\int_0^\infty x^{\rho(\beta-1)}\; e^{-2\rho\delta\left(\frac{x}{\theta}\right)^\beta}\left[\frac{2\lambda}{\lambda-1}-e^{\delta\left(\frac{x}{\theta}\right)^\beta}\right]^\rho dx\right\}.$$

Since, $\left[\frac{2\lambda}{\lambda-1}-e^{\delta\left(\frac{x}{\theta}\right)^\beta}\right]^\rho = \sum_{i=0}^{\rho}\binom{\rho}{i}\left(\frac{2\lambda}{\lambda-1}\right)^{\rho-i}\left(-e^{\delta\left(\frac{x}{\theta}\right)^\beta}\right)^i$, then

$$E_R(\rho) = \frac{1}{1-\rho}\log\left\{\left(\frac{\delta\beta(\lambda-1)}{\theta^\beta}\right)^\rho\int_0^\infty x^{\rho(\beta-1)}\; e^{-2\rho\delta\left(\frac{x}{\theta}\right)^\beta}\sum_{i=0}^{\rho}\binom{\rho}{i}\left(\frac{2\lambda}{\lambda-1}\right)^{\rho-i}\left(-e^{\delta\left(\frac{x}{\theta}\right)^\beta}\right)^i dx\right\}$$

$$= \frac{1}{1-\rho}\log\left\{\left(\frac{\delta\beta(\lambda-1)}{\theta^\beta}\right)^\rho\sum_{i=0}^{\rho}\binom{\rho}{i}\left(\frac{2\lambda}{\lambda-1}\right)^{\rho-i}(-1)^i\int_0^\infty x^{\rho(\beta-1)}\; e^{-\left(\frac{x}{\theta}\right)^\beta(2\rho\delta-i\delta)}dx\right\}.$$

Now, $\int_0^\infty x^{\rho(\beta-1)}\; e^{-\left(\frac{x^\beta}{\theta^\beta}\right)(2\rho\delta-i\delta)}dx = \int_0^\infty x^{\rho(\beta-1)}\; e^{-x^\beta\left(\frac{2\rho\delta-i\delta}{\theta^\beta}\right)}dx$.

Let $u = x^\beta\left(\frac{2\rho\delta-i\delta}{\theta^\beta}\right)$, $\frac{u}{\frac{2\rho\delta-i\delta}{\theta^\beta}} = x^\beta$, $\frac{\theta^\beta u}{2\rho\delta-i\delta} = x^\beta$, so $x = \sqrt[\beta]{\frac{\theta^\beta u}{2\rho\delta-i\delta}} = \frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}u^{\frac{1}{\beta}}$,

and hence $dx = \frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\frac{1}{\beta}u^{\frac{1}{\beta}-1}du$. Therefore,

$$\int_0^\infty x^{\rho(\beta-1)}\; e^{-x^\beta\left(\frac{2\rho\delta-i\delta}{\theta^\beta}\right)}dx = \int_0^\infty\left(\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}u^{\frac{1}{\beta}}\right)^{\rho(\beta-1)} e^{-u}\,\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\frac{1}{\beta}u^{\frac{1}{\beta}-1}du$$

$$= \int_0^\infty\left(\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}u^{\frac{1}{\beta}}\right)^{\rho(\beta-1)} e^{-u}\,\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\frac{1}{\beta}u^{\frac{1}{\beta}-1}du$$

$$= \frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\frac{1}{\beta}\left(\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\right)^{\rho(\beta-1)}\int_0^\infty u^{\frac{\rho(\beta-1)+1-\beta}{\beta}}e^{-u}du.$$

Thus,

$$E_R(\rho)=\frac{1}{1-\rho}\log\left\{\begin{array}{l}\left(\frac{\delta\beta(\lambda-1)}{\theta^{\beta}}\right)^{\rho}\sum_{i=0}^{\rho}\binom{\rho}{i}\left(\frac{2\lambda}{\lambda-1}\right)^{\rho-i}(-1)^{i}\\ \times\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\frac{1}{\beta}\left(\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\right)^{\rho(\beta-1)}\int_0^{\infty}u^{\frac{\rho(\beta-1)+1-\beta}{\beta}}e^{-u}du\end{array}\right\}$$

$$=\frac{1}{1-\rho}\log\left\{\left(\frac{\delta\beta(\lambda-1)}{\theta^{\beta}}\right)^{\rho}\sum_{i=0}^{\rho}\binom{\rho}{i}\left(\frac{2\lambda}{\lambda-1}\right)^{\rho-i}(-1)^{i}\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\frac{1}{\beta}\left(\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\right)^{\rho(\beta-1)}\Gamma\left(\frac{\rho(\beta-1)+1-\beta}{\beta}+1\right)\right\}$$

$$=\frac{1}{1-\rho}\left[\log\left\{\left(\frac{\delta\beta(\lambda-1)}{\theta^{\beta}}\right)^{\rho}\Gamma\left(\frac{\rho(\beta-1)+1}{\beta}\right)\sum_{i=0}^{\rho}\binom{\rho}{i}\left(\frac{2\lambda}{\lambda-1}\right)^{\rho-i}(-1)^{i}\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\frac{1}{\beta}\left(\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\right)^{\rho(\beta-1)}\right\}\right]$$

$$=\frac{1}{1-\rho}\left[\left\{\log\left(\frac{\delta\beta(\lambda-1)}{\theta^{\beta}}\right)^{\rho}+\log\Gamma\left(\frac{\rho(\beta-1)+1}{\beta}\right)+\log\sum_{i=0}^{\rho}\binom{\rho}{i}\left(\frac{2\lambda}{\lambda-1}\right)^{\rho-i}(-1)^{i}\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\frac{1}{\beta}\left(\frac{\theta}{\sqrt[\beta]{2\rho\delta-i\delta}}\right)^{\rho(\beta-1)}\right\}\right]$$

8. CONCLUSIONS

In this paper, the transmuted new Weibull-Pareto distribution is suggested, which is an extension of the new Weibull-Pareto distribution. The moments, the *r*th moment, the skewness, kurtosis, and the order statistics functions of the TNWP distribution are derived. Also, the maximum likelihood estimators are derived and we defined the hazard rate and reliability functions as well as Renyi entropy.

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REFERENCES

[1]. ALJARRAH, M.A., FAMOYE, F., and LEE, C. (2015). A New Weibull-Pareto distribution. **Communications in Statistics-Theory and Methods**, 44, 4077–4095.

[2]. AL-OMARI, A.I., AL-KHAZALEH, M. and ALZOUBI, L.M. (2017). Transmuted Janardan distribution: A Generalization of the Janardan distribution. **Journal of Statistics Applications and Probability**, 5: 1-11.

[3]. ARYAL, G.R. AND TSOKOS, C. (2009). On the transmuted extreme value distribution with application. **Nonlinear Analysis: Theory, Methods, & Applications**, 71, 1401-1407.

[4]. ELBATAL, I. and ARYAL, G. (2013). On the transmuted additive Weibull distribution. **Austrian Journal of Statistics**, 42, 117-132.

[5]. ELGARHY, M., RASHED, M. and SHAWKI, A.W. (2016). Transmuted generalized Lindley distribution**. International Journal of Mathematics Trends and Technology**, 29, 145-154.

[6]. KHAN, S.K. and KING, R. (2015). Transmuted modified inverse Rayleigh distribution. **Austrian Journal of Statistics**, 44, 17-29.

[7]. MEROVCI, F., ELBATAL, I. and AHMED, A. (2014). The transmuted generalized inverse Weibull distribution. **Austrian Journal of Statistics**, 43, 119-131.

[8]. NASIRU, S. and LUGUTERAH, A. (2015). The new Weibull-Pareto distribution. **Pakistan Journal of Statistics and Operation Research**, 11, 103-114.

[9]. SHAW, W. AND BUCKLEY, I. (2007). The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic normal distribution from a rank transmutation map," Research Report.

[10]. RÉNYI, A., (1961). On measures of entropy and information. University of California Press, Berkeley, California, 547–561.