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# THE CLASSICAL SUMUDU TRANSFORM AND ITS q- IMAGE INVOLVING MITTAG-LEFFLER FUNCTION

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**ABSTRACT**
The q-Sumudu transform, the q-image of classical Sumudu transform is the theoretical dual of the q-Laplace transform. It has a wide range of applications in quantum physics. In view of this, the present paper deals with some of the important applications of classical Sumudu and q-Sumudu transform of Mittag-Leffler function. The results have been presented in terms of well-known Fox's H-function. Some special cases have also been discussed.

**KEYWORDS:** Classical Sumudu transform, q-image of Sumudu transform, ML-Function.

**MSC:** 33D15, 44A10, 44A20.

**RESUMEN**
La q-Sumudu transformada, la q-imagen de la clásica transformada de Sumudu es el teorético dual l la q-Laplace transformada. Esto posee un amplio rango de aplicaciones en física quántica. Visto esto, este paper trata de algunas importantes aplicaciones de la transformada clásica de Sumudu y la q-Sumudu de la función de Mittag-Leffler. Los resultados han sido presentados en términos de la bien conocida Fox H-función. Algunos casos especiales han sido discutidos también.

**PALABRAS CLAVE:** transformada clásica de Sumudu, q-imagen de la transformada de Sumudu Clásica Sumudu, ML-Función.

## 1. INTRODUCTION:

The classical Laplace, Fourier, Mellin transforms and Sumudu transform have been widely used in mathematical physics and applied mathematics. The classical theory of the Laplace transform is well known [Sneddon (1951)] and its generalization was considered by many authors [Zemanian (1968), Rao (1974) and Saxena et al (1960,1961,1966)]. Various existence conditions and the detailed study about the range and invertibility was studied by Rooney (1982). The Mellin Transform and Sumudu transform which is the theoretical dual of the Laplace transform are widely used together to solve the fractional Kinetic equations, two–parameter fractional Telegraph equation and thermonuclear equations [Mathai and Haubold (2007)].

## 2. MATHEMATICAL PRELIMINARIES

**Classical Laplace transform:** Suppose F(t) is a real valued function defined over the interval $(0,\infty)$. The Laplace transform of F(t) is defined by

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt. \tag{1}$$

Or $f(s) = \int_0^\infty e^{-st} f(t)dt.$

The Laplace transform is said to exist if the integral (1) is convergent for some values of s.

**Classical Fourier transform:** If f(x) be a function defined on $(-\infty, \infty)$ uniformly continuous in finite interval and $\int_0^\infty \|f(x)\| d(x)$ converges. The Fourier transform is defined by
$F(f(x)) = \overline{f(s)} = \int_{-\infty}^{\infty} f(x) e^{isx} d(x)$, where $e^{isx}$ is said to be kernel of the Fourier transform.

**q- image of Laplace transform:** Hahn [2] defined the q-image of classical Laplace transform

$$L[f(t)] = \int_0^\infty e^{-st} f(t) dt. \tag{2}$$

by means of the following q-integrals

$$L_q f(s) = \int_0^\infty e_q^{-sx} f(x) d(x), \; Re(s) > 0 \tag{3}$$

Where $e_q^{-sx}$ is the q-analogue of exponential function [2].
The Laplace transform of the power function is defined as

$$L(t^\mu) = \frac{\Gamma(\mu+1)}{s^{\mu+1}} \tag{4}$$

The q-Laplace transform of the power function is defined as in [7 & 6]

$$L_q(t^\mu) = \frac{\Gamma_q(\mu+1)(1-q)^\mu}{s^{\mu+1}} \tag{5}$$

**Mittag-Leffler Function:**
The single parameter Mittag-Leffler function is defined by Mittag –Leffler [4], as follows:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\alpha n)}, \quad for \;\; \alpha \in C, R(\alpha) > 0 \tag{6}$$

Its generalization with two complex parameters was introduced by Wiman [15] as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta+\alpha n)}, \quad for \;\; \alpha, \beta \in C, R(\alpha) > 0 \tag{7}$$

In 1971, Prabhakar [5] introduced the generalized triple parameter ML function $E_{\alpha,\beta}^{\gamma}(z)$ as follows

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\beta+\alpha n)}, \quad for \;\; \alpha, \beta, \gamma \in C, R(\alpha) > 0 \tag{8}$$

Where, $(\gamma)_n = \gamma(\gamma+1)(\gamma+2)(\gamma+3) \ldots (\gamma+n-1)$ and $(\gamma)_0 = 1$.

**Classical Sumudu Transform:**

Over the set of function

$$A = \{f(t) | \exists M, \;\; \tau_1, \tau_2 > 0, |f(t)| < Me^{|t|/\tau_j}, \quad if \;\; t \in (-1)^j \times [0, \infty)\}$$

the Sumudu transform is defined by [14]

$$G(u) = S[f(t)] = \int_0^\infty f(ut) e^{-t} \, dt, \;\; u \in (-\tau_1, \tau_2) \tag{9}$$

**q-Image of Sumudu Transform:**
Albayrak, Purohit and Ucar [1] defined the $q$-analogues of the Sumudu transform by means of the following

$$S_{\mathbf{q}}\{f(t)\,; s\} = \frac{\mathbf{1}}{(\mathbf{1-q})\mathbf{s}} \int_{\mathbf{0}}^{\infty} \mathbf{E}_{\mathbf{q}}(\frac{\mathbf{q}}{\mathbf{s}}\mathbf{t}) f(t) \mathbf{d}_{\mathbf{q}}(\mathbf{t}). \tag{10}$$

$$\boldsymbol{S}_{\boldsymbol{q}}\{x^{\alpha-1}\,; s\} = s^{\alpha-1}(1-q)^{\alpha-1}\boldsymbol{\Gamma}_{\mathbf{q}}(\alpha) \tag{11}$$

Also, $(1-q)^{\alpha-1}\Gamma_q(\alpha) = (q;q)_{\alpha-1}$

## 3. CLASSICAL SUMUDU TRANSFORM OF ML-FUNCTION

In this section of paper, the authors have derived the classical Sumudu transform of generalized ML-Function $E_{\alpha,\beta}^{\gamma}(z)$ in terms of Fox's H – function. The corresponding theorem is given by:
**Theorem 3.1:** The classical Sumudu transform of generalized ML-Function $E_{\alpha,\beta}^{\gamma}(z)$ in terms of Fox's H – function is given by

$$S\left(E_{\alpha,\beta}^{\gamma}(z)\right) = \frac{1}{\Gamma(\gamma)} H_{1,1}^{1,1}\left[\begin{matrix}(1-\gamma,1) \\ (1-\beta,\alpha)\end{matrix} | u\right]$$

**Proof:** For $R(\alpha) > 0$, the classical Sumudu transform of ML-Function in terms of Fox's H – function is given by

$$S\left(E_{\alpha,\beta}^{\gamma}(z)\right) = S\left\{\sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\beta+\alpha n)} \frac{1}{n!}\right\} \tag{12}$$

Since, $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$.
From equation (12) we have,

$$S\left(E_{\alpha,\beta}^{\gamma}(z)\right) = S\left\{\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)z^n}{\Gamma(\beta+\alpha n)\Gamma(\gamma)}\frac{1}{n!}\right\}$$

Or $\qquad = \frac{1}{\Gamma(\gamma)}\left\{\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)}{\Gamma(\beta+\alpha n)}\frac{1}{\Gamma(n+1)}S(z^n)\right\}$

Or by (9) we get,

$S\left(E_{\alpha,\beta}^{\gamma}(z)\right) = \frac{1}{\Gamma(\gamma)}\left\{\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)}{\Gamma(\beta+\alpha n)}\frac{1}{\Gamma(n+1)}u^n\Gamma(n+1)\right\}$

Or $\qquad = \frac{1}{\Gamma(\gamma)}\left\{\sum_{n=0}^{\infty}\frac{\Gamma(1-(1-\gamma)+n)}{\Gamma(1-(1-\beta)+\alpha n)}u^n\right\} = \frac{1}{\Gamma(\gamma)}H_{1,1}^{1,1}\left[\begin{matrix}(1-\gamma,1)\\(1-\beta,\alpha)\end{matrix}|u\right]$

i.e., $S\left(E_{\alpha,\beta}^{\gamma}(z)\right) = \frac{1}{\Gamma(\gamma)}H_{1,1}^{1,1}\left[\begin{matrix}(1-\gamma,1)\\(1-\beta,\alpha)\end{matrix}|u\right]$. This completes the proof of theorem.

## 4. THE q-IMAGE OF SUMUDU TRANSFORM OF BASIC ANALOGUE OF ML-FUNCTION

In this section of paper, the authors have derived the q-image Sumudu transform of basic analogue ML-Function in terms of Fox's q-H – function which is given by

**Theorem 3.2:** The q- Sumudu transformof q- ML-Function in terms of q-H – function is given by

$$S_q\left(E_{\alpha,\beta}^{\gamma}(z;q)\right) = \frac{1}{\Gamma_q(\gamma)}H_{1,1}^{1,1}\left[\begin{matrix}(1-\gamma,1)\\(1-\beta,\alpha)\end{matrix}|u;q\right]$$

**Proof:** For $q > 0$, $R(\alpha) > 0$, the q-image of Sumudu transform of q-type of ML-Function in terms of basic analogue of H – function is given by

$$S_q\left(E_{\alpha,\beta}^{\gamma}(z;q)\right) = S_q\left\{\sum_{n=0}^{\infty}\frac{(\gamma;q)_n z^n}{\Gamma(\beta+\alpha n)}\frac{1}{(q;q)_n}\right\}$$

Or $\qquad = \left\{\sum_{n=0}^{\infty}\frac{\Gamma_q(\gamma+n)}{\Gamma_q(\beta+\alpha n)\Gamma_q(\gamma)}\frac{1}{(q;q)_n}\right\}S_q(z^n)$

By making use of (11) we get,

$S_q\left(E_{\alpha,\beta}^{\gamma}(z;q)\right) = \frac{1}{\Gamma_q(\gamma)}\left\{\sum_{n=0}^{\infty}\frac{\Gamma_q(\gamma+n)}{\Gamma_q(\beta+\alpha n)}\frac{1}{(q;q)_n}\right\}u^n(q;q)_n$

Or $\qquad = \frac{1}{\Gamma_q(\gamma)}\left\{\sum_{n=0}^{\infty}\frac{\Gamma_q(\gamma+n)}{\Gamma_q(\beta+\alpha n)}u^n\right\}$

Or $S_q\left(E_{\alpha,\beta}^{\gamma}(z;q)\right) = \frac{1}{\Gamma_q(\gamma)}\left\{\sum_{n=0}^{\infty}\frac{\Gamma_q(1-(1-\gamma)+n)}{\Gamma_q(1-(1-\beta)+\alpha n)}u^n\right\} = \frac{1}{\Gamma_q(\gamma)}H_{1,1}^{1,1}\left[\begin{matrix}(1-\gamma,1)\\(1-\beta,\alpha)\end{matrix}|u;q\right]$

i.e., $S_q\left(E_{\alpha,\beta}^{\gamma}(z;q)\right) = \frac{1}{\Gamma_q(\gamma)}H_{1,1}^{1,1}\left[\begin{matrix}(1-\gamma,1)\\(1-\beta,\alpha)\end{matrix}|u;q\right]$. This completes the proof.

**Observations**

**(3.1.1):** if $\gamma = 1$ then from above theorem $S_q\left(E_{\alpha,\beta}(z;q)\right) = H_{1,1}^{1,1}\left[\begin{matrix}(0,1)\\(1-\beta,\alpha)\end{matrix}|u;q\right]$

**(3.1.2):** if $\gamma = 1$ & $\beta = 1$ then from above theorem $S_q\left(E_{\alpha}(z;q)\right) = H_{1,1}^{1,1}\left[\begin{matrix}(0,1)\\(0,\alpha)\end{matrix}|u;q\right]$

**Special cases:**

Taking q = 1, we get following as special cases of theorem (3.2)

$$S\left(E_{\alpha,\beta}^{\gamma}(z)\right) = \frac{1}{\Gamma(\gamma)}H_{1,1}^{1,1}\left[\begin{matrix}(1-\gamma,1)\\(1-\beta,\alpha)\end{matrix}|u\right]$$

if $\gamma = 1$ then from above theorem

$$S\left(E_{\alpha,\beta}(z)\right) = H_{1,1}^{1,1}\left[\begin{matrix}(0,1)\\(1-\beta,\alpha)\end{matrix}|u\right]$$

And if $\gamma = 1$ & $\beta = 1$ then from above theorem

$$S\left(E_{\alpha}(z)\right) = H_{1,1}^{1,1}\left[\begin{matrix}(0,1)\\(0,\alpha)\end{matrix}|u\right].$$

## 5. CONCLUSION

The results proved in this paper give some contributions to the theory of the q- series, especially q- Bessel functions and may find applications to solutions of certain q-difference, q-integrals and q-transforms like Laplace and Fourier transforms.

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