BAYESIAN ESTIMATION UNDER KULLBACK-LEIBLER DIVERGENCE MEASURE BASED ON EXPONENTIAL DATA

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ABSTRACT
In information theory, Kullback-Leibler divergence measure is a commonly used difference measure that is used for computing the distance between two probability distributions. In this paper, we apply Kullback-Leibler divergence measure between actual and approximate distribution to drive a loss function. We then apply the derived loss function on Exponential distribution to find the Bayes estimate of the parameter θ, and compare it with the Bayes estimate obtained using square error loss function. Our comparisons between these two estimates are based on complete, type II censoring and type I censoring data.

KEYWORDS: Bayesian Estimation, Exponential distribution, Kullback-Leibler Divergence Measure, Complete Data, Type II Censored Data, Type I Censored Data, Maximum Likelihood Estimation.
MSC: 62N01, 62F15, 62F10, 94A17.

RESUMEN
En la teoría de la información la medida de divergencia de Kullback-Leibler se usa comunmente para medir la distancia entre dos distribuciones de probabilidad. En este trabajo se aplica la medida de divergencia de Kullback-Leibler entre la distribución real y la aproximada para derivar la función de pérdida. Se usa esta función en la distribución exponencial para hallar el estimador bayesiano del parámetro θ y lo comparamos con el estimador bayesiano calculado usando la funcón de pérdida usando el error cuadrático. Las comparaciones que se realizan entre ambos estimadores se basan en datos censurados de tipo I y tipo II.

PALABRAS CLAVE: Estimación Bayesiana, distribución exponencial, medida de divergencia de Kullback-Leiblerdatos completos, datos censurados de tipo I y de tipo II, estimación máximo verosímil

1. INTRODUCTION

Exponential distribution plays an important role in many situations of lifetime data analysis. In the last few years, many researchers have developed inference procedures for Exponential model. Sarhan

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(2003) obtained the empirical Bayes estimators of Exponential model. Janeen (2004) discussed the empirical Bayes estimators of the Exponential distribution parameter based on record values. For more details, one may refer to Balakrishnan et al. (2005).
Loss functions have a great importance in Bayesian analysis because they typically used in parameter estimation. The squared error loss and LINEX loss functions are unbounded loss functions and widely employed in decision theory due to its elegant mathematical properties, not its applicability to the representation of a true loss structure (Leon and Wu (1992)). In many real life situations, many examples illustrate that unbounded loss can be unduly restrictive and suggest that estimators based on a bounded loss function should be used instead. Moreover, the nature of many decision problems require the use of bounded loss functions, especially in financial problems. For more details see Berger (1985) or Kamińska and Porosiński (2009). To overcome the shortcoming of unbounded loss, several bounded loss functions were proposed by researchers. Lianwe et. al (2013) studied the Bayes estimation of the parameter of Exponential distribution under a bounded loss function, called reflected gamma loss function. Wen and Levy (2001) proposed a bounded asymmetric loss function called BLINEX loss function. Abufoudeh et. al (2016) studied the Bayesian inference of the Rayleigh parameters when the data are type II censored under the squared error loss function. Kundu and Howlader (2010) described the Bayesian inference and prediction of the inverse Weibull distribution for type II censored data, they obtained the Bayes estimators based on the square error loss function. Pradhan and Kundu (2011) studied the Bayesian estimation and prediction of the two parameter Gamma distribution, they considered the Bayes estimation under square error loss function with the assumption that the scale parameter has a Gamma prior and the shape parameter has any log-concave prior. Bdair and Raqab (2016) studied the Bayesian prediction problem for Weibull model under symmetric and asymmetric loss functions, they studied squared, absolute and Linex loss functions.

For parameters' estimation problem, many error loss functions are used by authors in the literature. One of the most famous among other is the square error loss function (SEL), defined as

$$L(\theta, \hat{\theta}) = \left(\theta - \hat{\theta}\right)^2$$

where $\hat{\theta}$ is the estimate of θ.
In this research work we make a comparison between the relative entropy measure as an alternative loss function and the square error loss function. The relative entropy measure or Kullback-Leibler divergence measure was first introduced by Kullback and Leibler (1951) which may be defined as follows:
Let $f(x)$ and $g(x)$ be two probability distributions for a continuous random variable X, the Kullback-Leibler (KL) divergence measure is defined as:

$$KL(f, g) = E_f[\log(\frac{f}{g})] = \int \log(\frac{f}{g}) f(x) dx, \tag{1.1}$$

which is the expectation of the logarithmic difference between the probabilities $f(x)$ and $g(x)$, where the expectation is taken with respect to the probability $f(x)$. The KL divergence is defined only if

$g(i) = 0$ implies $f(i) = 0$, for all i (absolute continuity). Unlike squared loss function, KL does not measure the discrepancy between an unknown parameter and its estimate, but between the unknown distribution f of the space $\underset{\sim}{X}$ and its estimate g. As a consequence, it is invariant with one-to-one reparametrization of the parameters and, hence, becomes a serious competitor to squared loss. Remark that KL is also invariant under one-to-one transformations of $\underset{\sim}{X}$ because such transforms do not affect the quantity of information carried by $\underset{\sim}{X}$.
An interesting property of the KL divergence that arises from Jensen's inequality is that $KL(f; g) \geq 0$ and $KL(f; g) = 0 \iff f = g \; almosteverywhere$.
Suppose that n components are put under test and their corresponding lifetimes $\underset{\sim}{X} = (X_1, X_2, ..., X_n)$ follow the Exponential distribution with the probability density function (pdf) given by:

$$f(x;\theta) = \begin{cases} \theta e^{-\theta x} & if \; x > 0, \theta > 0 \\ 0 & if \; x \leq 0 \; , \end{cases} \tag{1.2}$$

and cumulative distribution function (cdf)

$$F(x;\theta) = 1 - e^{-\theta x}, x > 0, \theta > 0. \tag{1.3}$$

Let $\hat{\theta}$ be an estimate of the Exponential distribution's parameter θ given in Eq. (2). We apply Eq. (1) to find the distance between the actual distribution $f(x;\theta)$ and the approximation distribution $\hat{f}(x;\hat{\theta})$ as

$$KL(f;\hat{f}) = E_f[\log(\frac{f}{\hat{f}})] = \frac{\hat{\theta}}{\theta} - \log(\frac{\hat{\theta}}{\theta}) - 1.$$

Let us call $KL(f;\hat{f})$ by Kullback error loss function (KEL), and denote it by $L_K(\theta;\hat{\theta})$.
In this article, we will apply KEL to find the Bayes estimate of the Exponential's parameter θ based on complete data, type II censored data and type I censored data that follow an Exponential distribution. The Bayesian estimation based on KEL function that is developed in this paper has many privileges over the well-known approaches in estimation like MLE and the regular Bayesian estimation approach. In fact, the Bayesian estimation under KEL is better than the MLE in the sense of the MSEs. On the other hand the Bayesian estimation under KEL is more efficient relative to the Bayesian estimation based on SEL function, the performances of its estimators in the sense of MSE are better than the ones obtained using SEL function when priors with variance greater than one are used. Our future works will concentrate on the comparisons between different types of loss functions like bounded, unbounded, symmetric and asymmetric with KEL when other general distributions are used.
The rest of this work is organized as follows: In Section 2, based on complete data set, we find the maximum likelihood estimation of θ, the Bayes estimate of θ under both SEL and KEL functions. In Sections 3 and 4, we present the maximum likelihood estimation and the Bayes estimations under SEL and KEL when type II and type I censoring data are used, respectively. Simulations of our results based on different generated schemes are presented in Section 5. In Section 6, a real life example is used to illustrate the so obtained results. Finally, we conclude our work in Section 7.

2. COMPLETE DATA

Let us consider a life-testing experiment where n items are kept under observation until failure. Suppose the life lengths of these n items are independent and identically distributed (iid) random variables with a common absolutely continuous cdf $F(x;\theta)$ and pdf $f(x;\theta)$ where θ is the unknown parameter. Then we have a random sample $X_1, X_2, ..., X_n$ from the cdf $F(x;\theta)$, these values are ordered in ascending order, that is , the data appear as the vector of order statistics in a natural way.

2.1. MAXIMUM LIKELIHOOD ESTIMATION

To find the maximum likelihood estimation (MLE) of the unknown parameter, we proceed as follows. Using Eq. (2), the well-known likelihood function is given by

$$\begin{aligned} L(\theta \mid data) &= \prod_{i=1}^{n} f(x_i \mid \theta) \\ &= \theta^n e^{-\theta \sum\limits_{i=0}^{n} x_i} \end{aligned}$$

After taking the natural logarithm and differentiate with respect to θ, MLE of θ, say $\hat{\theta}$, is

$$\hat{\theta} = \frac{n}{\sum\limits_{i=0}^{n} x_i} \tag{2.1}$$

2.2. BAYESIAN ESTIMATION

For the Bayesian inference, the posterior density of θ given the data is

$$\pi(\theta \mid data) \propto \prod_{i=1}^{n} f(x_i \mid \theta)\pi_1(\theta) \propto \theta^{n+a-1} e^{-\theta(\sum\limits_{i=0}^{n} x_i + b)}, \tag{2.2}$$

where $\pi_1(\theta)$ is the prior distribution of θ. The natural choice of the prior distribution is the conjugate gamma prior $Gamma\,(a, b)$, with pdf

$$\pi_1(\theta \mid a, b) = \begin{cases} \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} & if \quad \theta > 0 \\ 0 & if \quad \theta \leq 0. \end{cases}$$

Using Eq. (5), the posterior density of θ given the data is

$$Gamma(n + a,\ b + \sum_{i=0}^{n} x_i). \tag{2.3}$$

First, the Bayes estimate of θ under SEL function is the posterior mean, namely,

$$\hat{\theta}_{Bayes,S} = \frac{n + a}{b + \sum\limits_{i=0}^{n} x_i} \tag{2.4}$$

Second, to find the Bayes estimate of θ when KEL function is used, we need to find $\hat{\theta}$ which minimizing the risk function

$$E_\pi[L_K(\theta,\hat{\theta})] = \int_0^\infty \left(\frac{\hat{\theta}}{\theta} - \log\left(\frac{\overset{\wedge}{\theta}}{\theta}\right) - 1\right)\pi(\theta \mid data)d\theta. \tag{2.5}$$

Differentiating Eq. (8) with respect to $\hat{\theta}$ and equating the result by zero, the Bayes estimate of θ when KEL function is used, say $\hat{\theta}_{Bayes,K}$ is given by

$$\hat{\theta}_{Bayes,K} = \frac{1}{E_\pi\left(\frac{1}{\theta} \mid data\right)}. \tag{2.6}$$

Based on Eq. (6) and Eq. (9), the Bayes estimate of θ is

$$\hat{\theta}_{Bayes,K} = \frac{\Gamma(n+a)}{\left(b + \sum\limits_{i=0}^{r} x_i\right)\Gamma(n+a-1)}. \tag{2.7}$$

3. TYPE II CENSORED DATA

Let us consider a life-testing experiment where n items are kept under observation until failure. These items could be some systems, components or computer chips in some reliability experiment, or they could be patients put under certain drug or under some clinical conditions. Suppose the life lengths of these n items are iid random variables with a common absolutely continuous cdf $F(x;\theta)$ and pdf $f(x;\theta)$ where θ is the unknown parameter. Then we have a random sample $X_1, X_2, ..., X_n$ from the cdf $F(x;\theta)$. For some reason or other, one may terminate the experiment at the $r-th$ failure, that is, at time $X_{r:n}$. This situation is produced the type II censored sample $X_{1:n} < X_{2:n} < ... < X_{r:n}$. Here r is fixed $(1 < r \leq n)$, while $X_{r:n}$, the duration of the experiment is random. One may refer to David and Nagaraja (2003), Kundu and Raqab (2012) for more details.

3.1. MAXIMUM LIKELIHOOD ESTIMATION

In this subsection, we derive the MLE for the parameter θ of the Exponential model based on type II censored data. Let $X_{1:n} < X_{2:n} < ... < X_{r:n}$ be a type II censored sample of size r $(1 < r \leq n)$. The likelihood function is given by

$$L(\theta \mid x_1, x_2, ..., x_r) = \frac{n!}{(n-r)!}\prod_{i=1}^{r} f(x_i;\theta)\left[1 - F(x_r;\theta)\right]^{n-r}, \quad x_1 < x_2 < ... < x_r. \tag{3.1}$$

Using Eq.'s (2) and (3), the differentiation of the log-likelihood function with respect to θ yields that the MLE of θ $(\hat{\theta})$ is

$$\hat{\theta} = \frac{r}{\sum\limits_{i=0}^{r} x_i + (n-r)x_r} \tag{3.2}$$

3.2. BAYESIAN ESTIMATION

The posterior density of θ given the data is

$$\pi(\theta \mid data) \propto \frac{n!}{(n-r)!}\prod_{i=1}^{r} f(x_i \mid \theta)\,[1-F(x_r;\theta)]^{n-r}\,\pi_1(\theta) \propto \theta^{r+a-1} e^{-\theta(\sum\limits_{i=0}^{r} x_i+(n-r)x_r+b)}.$$

Therefore, the posterior density of θ given the data is

$$Gamma(r+a,\ b+(n-r)\,x_r+\sum_{i=0}^{r} x_i). \tag{3.3}$$

First, the Bayes estimate of θ under SEL function is the posterior mean, namely,

$$\hat{\theta}_{Bayes,S} = \frac{r+a}{b+(n-r)\,x_r+\sum\limits_{i=0}^{r} x_i}. \tag{3.4}$$

Second, using Eq. (13) and Eq. (9), the Bayes estimate of θ when KEL function is used, say $\hat{\theta}_{Bayes,K}$ is

$$\hat{\theta}_{Bayes,K} = \frac{\Gamma(r+a)}{\left(b+(n-r)\,x_r+\sum\limits_{i=0}^{r} x_i\right)\Gamma(r+a-1)}. \tag{3.5}$$

4. TYPE I CENSORED DATA

We know that in certain types of problems such as life-testing experiments, the ordered observations may occur naturally. In such cases, great savings in time and cost could be realized by terminating the experiment as soon as the first r ordered observations have occurred, rather than waiting for all n failures to occur. If one terminates the experiment after a fixed time t, this procedure is referred to as type I censored sampling. In this case the number of observation, R is a random variable. Suppose that out of n observations in total, only r observation occur before time t. Then given $R = r$,the likelihood function is given by

$$L(\theta \mid x_1, x_2, ..., x_r) = \frac{n!}{(n-r)!}\prod_{i=1}^{r} f(x_i;\theta)\,[1-F(t)]^{n-r}, x_1 < x_2 < ... < x_r, \tag{4.1}$$

where the probability that a failure occur before a time t for any given trial is $F(t)$, where F is the distribution function of the Exponential life-time model. For more details about type I censoring, one may refer to Gan and Bain (1998) or Joarder et. al (2011).

4.1. MAXIMUM LIKELIHOOD ESTIMATION

In this subsection, we derive the MLE for the parameter θ of the Exponential model based on type I censored data. Using Eq. (16), the differentiation of the log-likelihood function with respect to θ yields that the MLE of θ ($\hat{\theta}$) is

$$\hat{\theta} = \frac{r}{\sum_{i=0}^{r} x_i + (n-r)\,t}. \tag{4.2}$$

4.2. BAYESIAN ESTIMATION

The posterior density of θ given the data is

$$\pi(\theta \mid data) \propto \frac{n!}{(n-r)!} \prod_{i=1}^{r} f(x_i \mid \theta)\,[1-F(t)]^{n-r}\,\pi_1(\theta) \propto \theta^{r+a-1} e^{-\theta(\sum_{i=0}^{r} x_i + (n-r)t + b)}.$$

Therefore, the posterior density of θ given the data is

$$Gamma(r+a,\ b + (n-r)\,t + \sum_{i=0}^{r} x_i). \tag{4.3}$$

First, the Bayes estimate of θ under SEL function is the posterior mean, namely,

$$\hat{\theta}_{Bayes,S} = \frac{r+a}{b + (n-r)\,t + \sum_{i=0}^{r} x_i}. \tag{4.4}$$

Second, using Eq. (18) and Eq. (9), the Bayes estimate of θ when KEL function is used, say $\hat{\theta}_{Bayes,K}$ is

$$\hat{\theta}_{Bayes,K} = \frac{\Gamma(r+a)}{\left(b + (n-r)\,t + \sum_{i=0}^{r} x_i\right)\Gamma(r+a-1)}. \tag{4.5}$$

5. SIMULATIONS

In this simulation study, we report some numerical experiments performed to evaluate the behavior of the proposed methods for different sampling schemes and different priors based on complete, type II censored and type I censored Exponential data. We have assumed that $\theta = 2$ to generate the Exponential data. To compute the BEs under SEL function and KEL function, we have assumed $\pi_1(\theta)$, the prior of θ, has gamma density function with shape and scale parameters a and b, respectively. For the computations of BEs, we consider the following priors for θ: First prior is the non-informative prior, *i.e.* $a = b = 0$, we call this prior as Prior 0. Second prior is the informative prior, we use three informative priors according to the variance, when the variance is less than or equal to one, we use Prior 1 ($a = b = 1$) and Prior 3 ($a = 1, b = 2$). When the variance is greater than one, we use Prior 2 ($a = 2, b = 1$). In each sampling scheme, we compute the MLEs and the BEs of θ under SEL and KEL functions based on 10,000 samples. We report the average Bayes estimates and mean squared errors (MSEs) based on 10,000 replications. From Table 1, based on Exponential complete data, it can be noticed that the MLEs and Bayes estimates of θ improve when more information is available (n gets large) under SEL and KEL. This is valid for all suggested priors and we can also notice that the BEs for θ are better than the MLEs of θ under all priors. From Table 2, based on Exponential type II censored data, it can be noticed that the MLEs and Bayes estimates of θ improve when more information is available (r gets large) and the Bayes estimates of θ under the informative priors are much better than the non-informative prior according to the MSEs values under SEL and KEL. From Table 3 and Table 4, we notice that when t is increasing, the MLEs and Bayes estimates of θ under all priors are improved according to the MSEs values under SEL and KEL. The main results in this paper are as follows:

- The Bayes estimates of θ under KEL are better than the Bayes estimate of θ under SEL when the variance is greater than one, see for example Prior 2. This result is valid also for Prior 0.
- The Bayes estimates of θ under KEL is not better than the Bayes estimate of θ under SEL when the variance is less than or equal one, see Prior1 and Prior 3. This result is also valid for complete data, type II censored data and type I censored data.

Table 1: MLEs and Bayes estimates with respect to SEL function and KEL function based on Exponential complete data, when Prior 0, Prior 1, Prior 2 and Prior 3 are used.

Schemes				
	$n = 15$	$n = 20$	$n = 25$	$n = 30$
MLEs	2.1405 (0.3766)	2.1036 (0.2534)	2.0777 (0.1825)	2.0741 (0.1570)
Bayes				
Prior 0 (SEL)	2.1405 (0.3766)	2.1036 (0.2534)	2.0777 (0.1825)	2.0741 (0.1570)
Prior 0 (KEL)	2.0086 (0.3196)	1.9927 (0.2088)	1.9998 (0.1780)	1.9987 (0.1422)
Prior 1 (SEL)	1.9403(0.1655)	2.0121(0.1495)	1.9289(0.1361)	1.9664(0.1291)
Prior 1 (KEL)	1.8190(0.1675)	1.9162(0.1516)	1.8547(0.1423)	1.9030(0.1292)
Prior 2 (SEL)	2.0477(0.2298)	2.0701(0.1678)	2.0815(0.1389)	2.0573(0.1133)
Prior 2 (KEL)	1.9272(0.2068)	1.9760(0.1490)	2.0044(0.1227)	1.9930(0.1033)
Prior 3 (SEL)	1.7703(0.1631)	1.8377(0.1466)	1.9019(0.1212)	1.8615(0.1083)
Prior 3 (KEL)	1.6596(0.2128)	1.7502(0.1987)	1.8288(0.1324)	1.8014(0.1229)

Note: The first entry represents the point estimate, while the corresponding MSE is given between the parentheses.

Table 2: MLEs and Bayes estimates under SEL and KEL functions based on Exponential type II censored data, when Prior 0, Prior 1, Prior 2 and Prior 3 are used.

Schemes	$n = 15, r = 5$	$n = 20, r = 10$	$n = 25, r = 15$	$n = 30, r = 20$
MLEs	2.4878 (0.6811)	2.2147 (0.6610)	2.1441 (0.3748)	2.1018 (0.2632)
Bayes estimate				
Prior 0 (SEL)	2.4878 (0.6811)	2.2147 (0.6610)	2.1440 (0.3748)	2.1018 (0.2632)
Prior 0 (KEL)	1.9902 (0.5365)	1.9933 (0.4981)	2.0011 (0.3084)	1.9967 (0.2283)
Prior 1 (SEL)	1.9390(0.3529)	1.9776(0.2106)	1.9101(0.2246)	1.9350(0.1813)
Prior 1 (KEL)	1.6159(0.3900)	1.7978(0.2145)	1.7907(0.2341)	1.8429(0.1853)
Prior 2 (SEL)	2.0320(0.24512)	2.1507(0.2981)	2.0546(0.2438)	2.0638(0.1999)
Prior 2 (KEL)	1.7417(0.2384)	1.9714(0.2322)	1.9337(0.2177)	1.9700(0.1793)
Prior 3 (SEL)	1.5028(0.3419)	1.6321(0.2765)	1.7219(0.1960)	1.8348(0.1688)
Prior 3 (KEL)	1.2523(0.4247)	1.4837(0.3832)	1.6143(0.2531)	1.7474(0.1921)

Note: The first entry represents the point estimate, while the corresponding MSEs are given between the parentheses.

Table 3: MLEs and Bayes estimates of θ under SEL and KEL functions based on Exponential type I censored data, when Prior 0 and Prior 1 are used.

Scheme	MLEs	Bayes estimate (Prior 0, SEL)	Bayes estimate (Prior 0, KEL)	Bayes estimate (Prior 1, SEL)	Bayes estimate (Prior 1, KEL)
$n = 15, t = 0.5$	2.0917(0.4586)	2.0917(0.4586)	1.8758(0.4257)	1.8709(0.3634)	1.6937(0.4133)
$n = 15, t = 1.0$	2.1621(0.4879)	2.1621(0.4079)	1.9970(0.4115)	1.9271(0.2117)	1.7907(0.2307)
$n = 15, t = 1.5$	2.0757(0.3857)	2.0757(0.3857)	1.9301(0.3411)	1.9293(0.2020)	1.8028(0.2147)
$n = 20, t = 0.5$	2.1844(0.4816)	2.1844(0.4816)	2.0188(0.4160)	1.8682(0.2149)	1.7329(0.2577)
$n = 20, t = 1.0$	2.1410(0.3176)	2.1410(0.3176)	2.0193(0.2732)	1.9272(0.1535)	1.8227(0.1690)
$n = 20, t = 1.5$	2.0994(0.3164)	2.0994(0.3164)	1.9897(0.2792)	1.9450(0.1666)	1.8481(0.1232)
$n = 25, t = 0.5$	2.1259(0.3005)	2.1259(0.3005)	1.9962(0.2695)	1.9055(0.1972)	1.7924(0.2220)
$n = 25, t = 1.0$	2.1053(0.2065)	2.1053(0.2065)	2.0092(0.1828)	1.9416(0.1346)	1.8563(0.1439)
$n = 25, t = 1.5$	2.0675(0.1767)	2.0675(0.1767)	1.9808(0.1607)	1.8989(0.1108)	1.8220(0.1035)
$n = 30, t = 0.5$	2.1376(0.2904)	2.1376(0.2904)	2.0287(0.2599)	1.9132(0.1693)	1.8176(0.1886)
$n = 30, t = 1.0$	2.0614(0.2221)	2.0614(0.2221)	1.9825(0.1756)	1.9161(0.1561)	1.8441(0.1656)
$n = 30, t = 1.5$	2.0979(0.1853)	2.0979(0.1853)	2.0250(0.1660)	1.9603(0.1166)	1.8941(0.1198)

Note: The first entry represents the point estimate, while the corresponding MSEs are given between the parentheses.

Table 4: MLEs and Bayes of θ estimates under SEL and KEL functions based on Exponential type I censored data, when Prior 2 and Prior 3 are used.

Scheme	Bayes estimate (prior 2, SEL)	Bayes estimate (prior 2,KEL)	Bayes estimate (prior 3, SEL)	Bayes estimate (prior 3,KEL)
$n = 15, t = 0.5$	2.0953(0.3835)	1.9178(0.3531)	1.6023(0.3104)	1.4527(0.4423)
$n = 15, t = 1.0$	2.0763(0.1745)	1.9386(0.1567)	1.7548(0.2172)	1.6318(0.2781)
$n = 15, t = 1.5$	2.0757(0.1185)	1.9482(0.1034)	1.8080(0.2159)	1.6901(0.2555)
$n = 20, t = 0.5$	2.0984(0.2954)	1.9578(0.2702)	1.6103(0.3555)	1.4903(0.4526)
$n = 20, t = 1.0$	2.0568(0.1865)	1.9517(0.1726)	1.7617(0.1914)	1.6664(0.2364)
$n = 20, t = 1.5$	2.1260(0.1818)	2.0254(0.1455)	1.7797(0.1697)	1.6912(0.2066)
$n = 25, t = 0.5$	2.0618(0.2194)	1.9483(0.2077)	1.7384(0.1976)	1.6360(0.2556)
$n = 25, t = 1.0$	2.0659(0.1575)	1.9796(0.1448)	1.7805(0.1551)	1.7016(0.1899)
$n = 25, t = 1.5$	2.1207(0.1544)	2.0389(0.1102)	1.7732(0.1531)	1.7017(0.1840)
$n = 30, t = 0.5$	2.0472(0.1917)	1.9507(0.18413)	1.7117(0.1710)	1.6251(0.2254)
$n = 30, t = 1.0$	2.0541(0.1540)	1.9809(0.1437)	1.8511(0.1311)	1.7832(0.1507)
$n = 30, t = 1.5$	2.0552(0.0977)	1.9880(0.0896)	1.8795(0.1164)	1.8159(0.1300)

Note: The first entry represents the point estimate, while the corresponding MSEs are given between the parentheses.

6. REAL DATA ANALYSIS

In this section, we apply the results obtained in the above sections on a real life example. We analyze Exponential real lifetimes data (in minutes) to breakdown of an insulating fluid at voltage 40 kv. Such data are usually assumed to follow Exponential distribution in engineering theory and its applications. The real lifetimes data have been considered by Nelson (1982). The sample consists of twelve times ($n = 12$) to breakdown is as follows: $1, 1, 2, 3, 12, 25, 46, 56, 68, 109, 323, 417$. First we compute the

MLEs of θ that are found to be 1.1288, 2.1531 and 2.1150 based on complete data, type II censored data and type I censored data, respectively. Under Prior 2, the Bayes estimate of θ under KEL function is better than the Bayes estimate of θ under SEL function based on complete data, type II censored data and type I censored data. Under Prior 1 and Prior 3, the Bayes estimate of θ under KEL function is not better than the Bayes estimate of θ under SEL function based on complete data, type II censored data and type I censored data, see Table 5. As we noted in the previous section, the Bayes estimates under KLE function is still better than that under SEL function in the priors with variance greater than 1, while in priors with variance less than 1, the Bayes estimates under SEL is better than that under KLE function.

Table 5: MLEs and Bayes estimates of θ for real lifetimes data.

	Complete data	Type II censored data	Type I censored data
		$n = 9, r = 3$	$t = 60$ minutes
MLE	1.1288(0.7588)	2.1531(0.0934)	2.1150(0.1065)
		Bayes estimate	
Prior 1 (SEL)	1.1178(0.5782)	1.7374(0.0689)	1.9161(0.1561)
Prior 1 (KEL)	1.0318(0.6378)	1.9305(0.0848)	1.9431(0.1656)
Prior 2 (SEL)	2.1177(0.2318)	2.0320(0.2451)	2.1542(0.1340)
Prior 2 (KEL)	1.8972(0.1968)	1.6410(0.2194)	1.8819(0.1232)
Prior 3 (SEL)	1.9303(0.1831)	1.7351(0.2765)	1.7811(0.1201)
Prior 3 (KEL)	1.8096(0.2338)	1.3234(0.3132)	1.9832(0.1403)

7. CONCLUSION

Motivated by the importance of the estimation problem, various inferential methods have been discussed in the literature. We have proposed a Bayesian approach to estimate the distribution's parameter based on using KEL function. In Bayesian analysis, the so obtained estimator depends on the prior distribution and also on the error loss function. For this, our proposed priors are quite flexible in nature. Additionally, we have used two types of error loss functions to complete our comparison. In this context, this estimation problem is not considered previously in the literature. Our method of estimation was simple and quite useful since it allow us to assume priors with general set-up. In the sense of Bayesian estimation under KEL function, it is observed that it performs well when compared to the MLE and the Bayesian estimation under SEL function for priors with variance greater than one and for all suggested types of data. Moreover, it is evident that, the Bayesian estimation under KEL function is not very sensitive to the assumed values of the prior parameters.

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