AN EFFICIENT GENERALIZED FAMILY OF ESTIMATORS FOR MEAN ESTIMATION UNDER SIMPLE RANDOM SAMPLING

Usman Shahzad*, Muhammad Hanif*, Nursel Koyuncu**, Amelia Victoria Garcia Luengo***, Nadim Khan*

*Department of Mathematics and Statistics, PMAS-Arid Agriculture University, Rawalpindi, Pakistan.

**Department of Statistics, University of Hacettepe, Ankara, Turkey.

***Department of Mathematics, University of Almeria, Spain.

ABSTRACT

In this paper, we have envisaged an efficient generalized family of estimators for finite population mean of the study variable in simple random sampling (SRS) utilizing information on an supplementary variable. Asymptotic properties such as bias and mean square error (MSE) of the proposed generalized family of estimators have been determined. Further, we have shown that the proposed family of estimators is more efficient than reviewed estimators in literature. In the support of the theoretical proposed work we have given numerical illustration.

KEYWORDS: Bias, Mean Square Error (MSE), Auxiliary/Supplementary Variable, Percentage Relative Efficiency (PRE).

MSC: 62D05

RESUMEN

En este trabajo, hemos propuesto una familia generalizada de estimadores eficiente para estimar la media poblacional finita de la variable objeto de estudio que utiliza informacin sobre una variable complementaria bajo muestreo aleatorio simple (MAS). Se han determinado propiedades asintticas como el sesgo y el error cuadrtico medio (ECM) de la familia de estimadores generalizada propuesta. Adems, hemos demostrado que la familia de estimadores propuesta es ms eficiente que los estimadores revisados en la literatura. Como apoyo el trabajo terico propuesto, hemos dado una ilustracin numrica.

PALABRAS CLAVE: sesgo, error cuadrtico medio (ECM), variable auxiliar / complementaria, porcentaje de eficiencia relativa (PER).

1. INTRODUCTION AND PRELEMINARIES

In survey sampling, literature expresses a number of techniques for utilizing auxiliary information. The ratio, regression and product methods of estimation are widely used illustrations in this context. When some parameters of the auxiliary variable X have been known such as coefficient of variation

*Email of the corresponding author

(C_x), coefficient of kurtosis $(\beta_2(x))$, mean $(\bar{X})$, coefficient of skewness $(\beta_1(x))$ and the coefficient of correlation (ρ) between the study variable Y and X, several authors assessed large number of estimators/classes of estimators for the population mean $\bar{Y}$ of the study variable (see [6], [21], [19], [5], [20], [27], [31], [24], [34], [16], [25], [26], [17], [8], [9], [22], [33], [23], [15], [32],[11], [1], [2],[3] and [4]). These modifications allow researchers to utilize the type of information $(C_x, \beta_2(x), \bar{X}, \beta_1(x))$ which collected from the auxiliary variable X for efficient estimation of population mean $\bar{Y}$ in simple random sampling.

Suppose Y be the variate of interest and X be the auxiliary variate defined on a finite population U containing 1 to N units. Let $\bar{y} = \dfrac{\sum_{i=1}^{n} y_i}{n}$ and $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$ be the sample means of the study and auxiliary variables respectively having size n. Let us define $e_o = \frac{\bar{y}-\bar{Y}}{\bar{Y}}$ and $e_1 = \frac{\bar{x}-\bar{X}}{\bar{X}}$. The expectation of these e terms under simple random sampling design can be written as:

$$\begin{aligned}
E(e_o) &= E(e_1) = 0,\\
E(e_o^2) &= \left(\frac{1-f}{n}\right) C_y^2 = \mathfrak{g}_o,\\
E(e_1^2) &= \left(\frac{1-f}{n}\right) C_x^2 = \mathfrak{g}_1,\\
E(e_o e_1) &= \left(\frac{1-f}{n}\right) \rho C_y C_x = \mathfrak{g}_{o1}.
\end{aligned}$$

where $C_z = \frac{S_z}{\bar{Z}}$ is the coefficient of variation and S_z is the standart deviation of variables (z=y,x).

In this study, our objective is to construct the generalized family of estimators utilizing single auxiliary variable under simple random sampling scheme which covers a number of estimators available in literature, also provides more efficient results as compare to existing ones. The properties such as bias and mean squared error (MSE) of the proposed estimator are derived under first order of approximation.

The rest of the study is as follows: in Section 2, we give several existing estimators for the finite population mean. The proposed generalized family of estimators is given in Section 3 along with its properties such as bias and MSE expressions. In Section 4, we provide efficiency comparisons to assess the performances of the proposed and existing estimators. The numerical illustration is given in Section 5, and conclusion is given in Section 6.

2. REVIEWED ESTIMATORS OF THE POPULATION MEAN

A huge amount of literature has been developed on survey sampling for addressing the issue of improved estimation of mean of study variable when information on supplementary/auxiliary variables is available. Our study is limited to simple random sampling scheme without replacement (SRSWOR). Some famous estimators in this field are as follows:

The variance of unbiased estimator $\bar{y}$ is given by

$$V(\bar{y}) = \bar{Y}^2 \mathfrak{g}_o. \tag{2.1}$$

[6] developed the following mean per unit estimator as follows

$$\hat{\mathfrak{T}}_r = \bar{y}\left[\frac{\bar{X}}{\bar{x}}\right]. \tag{2.2}$$

The bias and MSE of $\hat{\mathfrak{T}}_r$ are given by

$$\mathcal{B}(\hat{\mathfrak{T}}_r) = \bar{Y}\left[\mathfrak{g}_1 - \mathfrak{g}_{o1}\right].$$

$$\mathcal{MSE}(\hat{\mathfrak{T}}_r) = \bar{Y}^2\left[\mathfrak{g}_o + \mathfrak{g}_1 - 2\mathfrak{g}_{o1}\right]. \tag{2.3}$$

The conventional regression estimator is given by

$$\hat{\mathfrak{T}}_{reg} = \bar{y} + b\left(\bar{X} - \bar{x}\right). \tag{2.4}$$

where b is the sample regression coefficient.
The MSE of $\hat{\mathfrak{T}}_{reg}$ is

$$\mathcal{MSE}(\hat{\mathfrak{T}}_{reg}) = \bar{Y}^2\left[\mathfrak{g}_o - \frac{\mathfrak{g}_{01}^2}{\mathfrak{g}_1}\right]. \tag{2.5}$$

[5] developed the following exponential estimator

$$\hat{\mathfrak{T}}_{bt} = \bar{y}exp\left[\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right]. \tag{2.6}$$

The bias and MSE of $\hat{\mathfrak{T}}_{bt}$ are

$$\mathcal{B}(\hat{\mathfrak{T}}_{bt}) = \bar{Y}\left[\frac{3}{8}\mathfrak{g}_1 - \frac{1}{2}\mathfrak{g}_{o1}\right]$$

$$\mathcal{MSE}(\hat{\mathfrak{T}}_{bt}) = \bar{Y}^2\left[\mathfrak{g}_o + \frac{\mathfrak{g}_1}{4} - \mathfrak{g}_{o1}\right]. \tag{2.7}$$

[20] introduced the following estimator for the estimation of population mean as follows

$$\hat{\mathfrak{T}}_{rao} = \left[\mathfrak{q}_{rao1}\bar{y} + \mathfrak{q}_{rao2}\left(\bar{X} - \bar{x}\right)\right] \tag{2.8}$$

where $\mathfrak{q}_{rao1}$ and $\mathfrak{q}_{rao2}$ are scalars. The bias and MSE of $\hat{\mathfrak{T}}_{rao}$ are

$$\mathcal{B}(\hat{\mathfrak{T}}_{rao}) = \bar{Y}[\mathfrak{q}_{rao1} - 1],$$

$$\mathcal{MSE}(\hat{\mathfrak{T}}_{rao}) = \bar{Y}^2 + \mathfrak{q}_{rao1}^2\Theta_{A_{rao}} + \mathfrak{q}_{rao2}^2\Theta_{B_{rao}} + 2\mathfrak{q}_{rao1}\mathfrak{q}_{rao2}\Theta_{C_{rao}} - 2\mathfrak{q}_{rao1}\Theta_{D_{rao}},$$

where

$$\Theta_{A_{rao}} = \bar{Y}^2\left[1 + \mathfrak{g}_o\right],$$
$$\Theta_{B_{rao}} = \bar{X}^2\mathfrak{g}_1,$$
$$\Theta_{C_{rao}} = \bar{X}\bar{Y}\mathfrak{g}_{o1},$$
$$\Theta_{D_{rao}} = \bar{Y}^2.$$

Which is minimum for

$$\mathfrak{q}_{rao1}^{opt} = \left[\frac{\Theta_{B_{rao}} \Theta_{D_{rao}}}{\Theta_{A_{rao}} \Theta_{B_{rao}} - \Theta_{C_{rao}}^2} \right],$$

and

$$\mathfrak{q}_{rao2}^{opt} = \left[- \frac{\Theta_{C_{rao}} \Theta_{D_{rao}}}{\Theta_{A_{rao}} \Theta_{B_{rao}} - \Theta_{C_{rao}}^2} \right].$$

The minimum MSE of $\hat{\mathfrak{T}}_{rao}$ is given by

$$\mathcal{MSE}_{min}(\hat{\mathfrak{T}}_{rao}) = \left[\bar{Y}^2 - \frac{\Theta_{B_{rao}} \Theta_{D_{rao}}^2}{\Theta_{A_{rao}} \Theta_{B_{rao}} - \Theta_{C_{rao}}^2} \right]. \tag{2.9}$$

Some authors have defined following form of estimators

$$\hat{\mathfrak{T}}_i = \psi \lambda_i, \quad for\ i = 1, 2, ..., 26, \tag{2.10}$$

where $\psi = \left[\bar{y} + b(\bar{X} - \bar{x}) \right]$, $\gamma_i = \frac{c\bar{Y}}{c\bar{X}+d}$ and $\lambda_i = \frac{c\bar{X}+d}{c\bar{x}+d}$.
[12] $(for\ i = 1, 2, ..., 5)$ developed a class of ratio estimators by using the conventional descriptives of population. After that a number of researchers adopted their strategy and provided several modified ratio estimators (c.f., [13]; $(for\ i = 6, 7, ..., 10)$, [34]; $(for\ i = 11, 12)$, ([28], [29], [30]); $(for\ i = 13, 14, ..., 16)$, [10]; $(for\ i = 17)$ and [1]; $(for\ i = 18, 19, ..., 26)$) are given in Table 1.

Table 1: Estimators of [12] and those based on their adaptation

Est.	c	d	Est.	c	d
$\hat{\mathfrak{T}}_1 = \psi\lambda_1$	1	0	$\hat{\mathfrak{T}}_{14} = \psi\lambda_1$	C_x	M_d
$\hat{\mathfrak{T}}_2 = \psi\lambda_1$	1	C_x	$\hat{\mathfrak{T}}_{15} = \psi\lambda_1$	$\beta_1(x)$	M_d
$\hat{\mathfrak{T}}_3 = \psi\lambda_1$	1	$\beta_2(x)$	$\hat{\mathfrak{T}}_{16} = \psi\lambda_1$	$\beta_2(x)$	M_d
$\hat{\mathfrak{T}}_4 = \psi\lambda_1$	$\beta_2(x)$	C_x	$\hat{\mathfrak{T}}_{17} = \psi\lambda_1$	$\beta_1(x)$	$Q.D$
$\hat{\mathfrak{T}}_5 = \psi\lambda_1$	C_x	$\beta_2(x)$	$\hat{\mathfrak{T}}_{18} = \psi\lambda_1$	1	TM
$\hat{\mathfrak{T}}_6 = \psi\lambda_1$	1	ρ	$\hat{\mathfrak{T}}_{19} = \psi\lambda_1$	C_x	TM
$\hat{\mathfrak{T}}_7 = \psi\lambda_1$	C_x	ρ	$\hat{\mathfrak{T}}_{20} = \psi\lambda_1$	ρ	TM
$\hat{\mathfrak{T}}_8 = \psi\lambda_1$	ρ	C_x	$\hat{\mathfrak{T}}_{21} = \psi\lambda_1$	1	MR
$\hat{\mathfrak{T}}_9 = \psi\lambda_1$	$\beta_2(x)$	ρ	$\hat{\mathfrak{T}}_{22} = \psi\lambda_1$	C_x	MR
$\hat{\mathfrak{T}}_{10} = \psi\lambda_1$	ρ	$\beta_2(x)$	$\hat{\mathfrak{T}}_{23} = \psi\lambda_1$	ρ	MR
$\hat{\mathfrak{T}}_{11} = \psi\lambda_1$	1	$\beta_1(x)$	$\hat{\mathfrak{T}}_{24} = \psi\lambda_1$	1	HL
$\hat{\mathfrak{T}}_{12} = \psi\lambda_1$	$\beta_1(x)$	$\beta_2(x)$	$\hat{\mathfrak{T}}_{25} = \psi\lambda_1$	C_x	HL
$\hat{\mathfrak{T}}_{13} = \psi\lambda_1$	1	M_d	$\hat{\mathfrak{T}}_{26} = \psi\lambda_1$	ρ	HL

where M_d, $Q.D$, TM, MR and HL are median, quartile deviation, tri-mean, mid-range and Hodges-Lehmann estimator, respectively. For the details about the terms mentioned in above table see [1].
The MSE of $\hat{\mathfrak{T}}_i$ $(for\ i = 1, 2, .., 26)$, is given by

$$MSE(\hat{\mathfrak{T}}_i) = f' \left[\gamma_i^2 S_x^2 + S_y^2 (1 - \rho^2) \right]. \tag{2.11}$$

[14] proposed four new estimators by merging the estimators of [12]. These estimators are given by

$$\hat{\mathfrak{T}}_{KC1} = w_1^{KC1}\ \frac{\bar{y}+b(\bar{X}-\bar{x})}{\bar{x}}\bar{X} + w_2^{KC1}\ \frac{\bar{y}+b(\bar{X}-\bar{x})}{\bar{x}+C_x}(\bar{X}+C_x), \tag{2.12}$$

$$\hat{\mathfrak{T}}_{KC2} = w_1^{KC2}\ \frac{\bar{y}+b(\bar{X}-\bar{x})}{\bar{x}}\bar{X} + w_2^{KC2}\ \frac{\bar{y}+b(\bar{X}-\bar{x})}{\bar{x}+\beta_2(x)}(\bar{X}+\beta_2(x)), \tag{2.13}$$

$$\hat{\mathfrak{T}}_{KC3} = w_1^{KC3}\ \frac{\bar{y}+b(\bar{X}-\bar{x})}{\bar{x}}\bar{X} + w_2^{KC3}\ \frac{\bar{y}+b(\bar{X}-\bar{x})}{\bar{x}\beta_2(x)+C_x}(\bar{X}\beta_2(x)+C_x), \tag{2.14}$$

$$\hat{\mathfrak{T}}_{KC4} = w_1^{KC4}\ \frac{\bar{y}+b(\bar{X}-\bar{x})}{\bar{x}}\bar{X} + w_2^{KC4}\ \frac{\bar{y}+b(\bar{X}-\bar{x})}{C_x\bar{x}+\beta_2(x)}(C_x\bar{X}+\beta_2(x)), \tag{2.15}$$

where optimum values of w_1^{KCi} and w_2^{KCi} for $(i = 1, 2, 3, 4)$ are

$$w_1^{KC1(opt)} = \frac{\gamma_{10}}{\gamma_{10}-\gamma_9}, w_2^{KC1(opt)} = \frac{\gamma_9}{\gamma_9-\gamma_{10}}, w_1^{KC2(opt)} = \frac{\gamma_{11}}{\gamma_{11}-\gamma_9}, w_2^{KC2(opt)} = \frac{\gamma_9}{\gamma_9-\gamma_{11}},$$

$$w_1^{KC3(opt)} = \frac{\gamma_{12}}{\gamma_{12}-\gamma_9}, w_2^{KC3(opt)} = \frac{\gamma_9}{\gamma_9-\gamma_{12}}, w_1^{KC4(opt)} = \frac{\gamma_{13}}{\gamma_{13}-\gamma_9}, w_2^{KC4(opt)} = \frac{\gamma_9}{\gamma_9-\gamma_{13}}.$$

The MSE minimum of $\hat{\mathfrak{T}}_{KCi}$ $(i = 1, 2, 3, 4)$ is

$$MSE_{min}(\hat{\mathfrak{T}}_{KCi}) = MSE(\hat{\mathfrak{T}}_{Reg}). \tag{2.16}$$

Similarly, by taking motivation from [14] one can merge any two estimators $\hat{\mathfrak{T}}_i$ $(for\ i = 1, 2, .., 26)$ and get a result that minimum MSEs of merged estimators are equal to MSE of regression estimator. [16] constructed a family of estimators as follows

$$\hat{\mathfrak{T}}_{kr} = \bar{y}\left[\frac{\left(a\bar{X}+b\right)}{\alpha\left(a\bar{x}+b\right)+(1-\alpha)\left(a\bar{X}+b\right)}\right]^g. \tag{2.17}$$

The bias and MSE of $\hat{\mathfrak{T}}_{kr}$ are

$$\mathcal{B}(\hat{\mathfrak{T}}_{kr}) = \bar{Y}\left[\frac{g(g+1)}{2}\alpha^2\mathfrak{v}^2\mathfrak{g}_1 - g\alpha\mathfrak{v}\mathfrak{g}_{o1}\right],$$

$$\mathcal{MSE}(\hat{\mathfrak{T}}_{kr}) = \bar{Y}^2\left[\mathfrak{g}_o + \alpha^2\mathfrak{v}^2g^2\mathfrak{g}_1 - 2\alpha\mathfrak{v}g\mathfrak{g}_{o1}\right]$$

where $\mathfrak{v} = \dfrac{a\bar{X}}{a\bar{X}+b}$.

The minimum MSE of $\hat{\mathfrak{T}}_{kr}$ is equal to MSE of $\hat{\mathfrak{T}}_{reg}$ for $\alpha^{opt} = \dfrac{\mathfrak{g}_{o1}}{\mathfrak{v}g\mathfrak{g}_1}$.

Note that for $\alpha = 1$ and replacing (a, b) with different known population characteristics; $\hat{\mathfrak{T}}_{kr}$ can be converted into [27], [31], [24], [34], [28], [29], [30], [10] estimators.

[17] extend the work of [16] and develop an efficient family of estimators as follows

$$\hat{\mathfrak{T}}_{kk} = w\bar{y}\left[\frac{\left(\bar{X}a+b\right)}{\alpha\left(\bar{x}a+b\right)+(1-\alpha)\left(\bar{X}a+b\right)}\right]^g. \tag{2.18}$$

Note that [21] and [19] estimators are also the member of [17] family, denoted by $\hat{\mathfrak{T}}_{kk_1}$ and $\hat{\mathfrak{T}}_{kk_2}$ respectively in Table [2].

The bias and MSE of $\hat{\mathfrak{T}}_{kk}$ are given by

$$\mathcal{B}(\hat{\mathfrak{T}}_{kk}) = w\bar{Y}\left[\frac{g(g+1)}{2}\alpha^2\mathfrak{v}^2\mathfrak{g}_1 - g\alpha\mathfrak{v}\mathfrak{g}_{o1}\right] + \bar{Y}(w-1),$$

$$\mathcal{MSE}(\hat{\mathfrak{T}}_{kk}) = \bar{Y}^2 + w^2\Theta_{B_{kk}} - 2w\Theta_{A_{kk}}$$

where

$$\Theta_{A_{kk}} = \bar{Y}^2\left[1 + \frac{g(g+1)}{2}\alpha^2\mathfrak{v}^2\mathfrak{g}_1 - g\alpha\mathfrak{v}\mathfrak{g}_{o1}\right],$$
$$\Theta_{B_{kk}} = \bar{Y}^2\left[1 + \mathfrak{g}_o + \left(2g^2+g\right)\alpha^2\mathfrak{v}^2\mathfrak{g}_1 - 4g\alpha\mathfrak{v}\mathfrak{g}_{o1}\right].$$

Which is minimum for

$$w^{opt} = \frac{\Theta_{A_{kk}}}{\Theta_{B_{kk}}}.$$

The minimum MSE of $\hat{\mathfrak{T}}_{kk}$ is given by

$$\mathcal{MSE}_{min}(\hat{\mathfrak{T}}_{kk}) = \left[\bar{Y}^2 - \frac{\Theta_{A^2_{kk}}}{\Theta_{B_{kk}}}\right]. \tag{2.19}$$

Table 2: Family members of [17] for ($\alpha = 1$)

Est.	a	b
$\hat{\mathfrak{T}}_{kk_1} = w\bar{y}$	0	1
$\hat{\mathfrak{T}}_{kk_2} = w\bar{y}\left[\frac{\bar{X}}{\bar{x}}\right]$	1	0
$\hat{\mathfrak{T}}_{kk_3} = w\bar{y}\left[\frac{\bar{X}+C_x}{\bar{x}+C_x}\right]$	1	C_x
$\hat{\mathfrak{T}}_{kk_4} = w\bar{y}\left[\frac{\bar{X}\beta_2(x)+C_x}{\bar{x}\beta_2(x)+C_x}\right]$	$\beta_2(x)$	C_x
$\hat{\mathfrak{T}}_{kk_5} = w\bar{y}\left[\frac{\bar{X}C_x+\beta_2(x)}{\bar{x}C_x+\beta_2(x)}\right]$	C_x	$\beta_2(x)$
$\hat{\mathfrak{T}}_{kk_6} = w\bar{y}\left[\frac{\bar{X}+S_x}{\bar{x}+S_x}\right]$	1	S_x
$\hat{\mathfrak{T}}_{kk_7} = w\bar{y}\left[\frac{\bar{X}\beta_1(x)+S_x}{\bar{x}\beta_1(x)+S_x}\right]$	$\beta_1(x)$	S_x
$\hat{\mathfrak{T}}_{kk_8} = w\bar{y}\left[\frac{\bar{X}\beta_2(x)+S_x}{\bar{x}\beta_2(x)+S_x}\right]$	$\beta_2(x)$	S_x
$\hat{\mathfrak{T}}_{kk_9} = w\bar{y}\left[\frac{\bar{X}+\rho}{\bar{x}+\rho}\right]$	1	ρ
$\hat{\mathfrak{T}}_{kk_{10}} = w\bar{y}\left[\frac{\bar{X}+\beta_2(x)}{\bar{x}+\beta_2(x)}\right]$	1	$\beta_2(x)$

[8] introduced the following exponential estimator for the estimation of population mean as follows

$$\hat{\mathfrak{T}}_{lp} = \left[\mathfrak{q}_{lp1}\bar{y} + \mathfrak{q}_{lp2}\left(\bar{X}-\bar{x}\right)\right] exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right]. \tag{2.20}$$

The bias and MSE of $\hat{\mathfrak{T}}_{lp}$ are given by

$$\mathcal{B}(\hat{\mathfrak{T}}_{lp}) = \mathfrak{q}_{lp1}\bar{Y}\left[1+\frac{3}{8}\mathfrak{g}_1 - \frac{1}{2}\mathfrak{g}_{o1}\right] + \mathfrak{q}_{lp2}\bar{X}\frac{\mathfrak{g}_1}{2} - \bar{Y},$$

$$\mathcal{MSE}(\hat{\mathfrak{T}}_{lp}) = \bar{Y}^2 + \mathfrak{q}_{lp1}^2\Theta_{A_{lp}} + \mathfrak{q}_{lp2}^2\Theta_{B_{lp}} + 2\mathfrak{q}_{lp1}\mathfrak{q}_{lp2}\Theta_{C_{lp}} - 2\mathfrak{q}_{lp1}\Theta_{D_{lp}} - 2\mathfrak{q}_{lp2}\Theta_{E_{lp}}$$

where

$$\begin{aligned}
\Theta_{A_{lp}} &= \bar{Y}^2\left[1+\mathfrak{g}_o+\mathfrak{g}_1-2\mathfrak{g}_{o1}\right],\\
\Theta_{B_{lp}} &= \bar{X}^2\mathfrak{g}_1,\\
\Theta_{C_{lp}} &= \bar{X}\bar{Y}\left[\mathfrak{g}_1-\mathfrak{g}_{o1}\right],\\
\Theta_{D_{lp}} &= \bar{Y}^2\left[1+\frac{3\mathfrak{g}_1}{8}-\frac{\mathfrak{g}_{o1}}{2}\right],\\
\Theta_{E_{lp}} &= \bar{X}\bar{Y}\frac{\mathfrak{g}_1}{2}.
\end{aligned}$$

Which is minimum for

$$\mathfrak{q}_{lp1}^{opt} = \left[\frac{\Theta_{B_{lp}}\Theta_{D_{lp}} - \Theta_{C_{lp}}\Theta_{E_{lp}}}{\Theta_{A_{lp}}\Theta_{B_{lp}} - \Theta_{C_{lp}}^2}\right],$$

and

$$\mathfrak{q}_{lp2}^{opt} = \left[\frac{\Theta_{A_{lp}}\Theta_{E_{lp}} - \Theta_{C_{lp}}\Theta_{D_{lp}}}{\Theta_{A_{lp}}\Theta_{B_{lp}} - \Theta_{C_{lp}}^2}\right].$$

The minimum MSE of $\hat{\mathfrak{T}}_{lp}$ is given by

$$\mathcal{MSE}_{min}(\hat{\mathfrak{T}}_{lp}) = \left[\bar{Y}^2 - \frac{\Theta_{B_{lp}}\Theta_{D_{lp}}^2 + \Theta_{A_{lp}}\Theta_{E_{lp}}^2 - 2\Theta_{C_{lp}}\Theta_{D_{lp}}\Theta_{E_{lp}}}{\Theta_{A_{lp}}\Theta_{B_{lp}} - \Theta_{C_{lp}}^2}\right]. \tag{2.21}$$

[22] introduced the following estimator for the estimation of population mean as follows

$$\hat{\mathfrak{T}}_{sg} = \left[\mathfrak{q}_{sg1}\bar{y} + \mathfrak{q}_{sg2}\left(\bar{X}-\bar{x}\right)\right] exp\left[\frac{A^{'}-a^{'}}{A^{'}+a^{'}}\right] \tag{2.22}$$

where $A^{'} = \bar{X} + N\bar{X}$ and $a^{'} = \bar{x} + N\bar{X}$.
By substituting $A^{'}$ and $a^{'}$, we can write $\hat{\mathfrak{T}}_{sg}$ as

$$\hat{\mathfrak{T}}_{sg} = \left[\mathfrak{q}_{sg1}\bar{y} + \mathfrak{q}_{sg2}\left(\bar{X}-\bar{x}\right)\right] exp\left[\frac{\bar{X}-\bar{x}}{2N\bar{X}+\bar{X}+\bar{x}}\right].$$

The bias and MSE of $\hat{\mathfrak{T}}_{sg}$ are given by

$$\mathcal{B}(\hat{\mathfrak{T}}_{sg}) = \bar{Y}\left[(\mathfrak{q}_{sg1}-1) + \mathfrak{q}_{sg1}\left\{\frac{3\mathfrak{g}_1}{8(1+N)^2} - \frac{\mathfrak{g}_{o1}}{2(1+N)}\right\}\right] + \mathfrak{q}_{sg2}\bar{X}\frac{\mathfrak{g}_1}{2(1+N)^2},$$

$$\mathcal{MSE}(\hat{\mathfrak{T}}_{sg}) = \bar{Y}^2 + \mathfrak{q}_{sg1}^2\Theta_{A_{sg}} + \mathfrak{q}_{sg2}^2\Theta_{B_{sg}} + 2\mathfrak{q}_{sg1}\mathfrak{q}_{sg2}\Theta_{C_{sg}} - 2\mathfrak{q}_{sg1}\Theta_{D_{sg}} - 2\mathfrak{q}_{sg2}\Theta_{E_{sg}},$$

where

$$\Theta_{A_{sg}} = \bar{Y}^2\left[1+\mathfrak{g}_o+\frac{\mathfrak{g}_1}{(N+1)^2}-\frac{2\mathfrak{g}_{o1}}{(N+1)}\right],$$
$$\Theta_{B_{sg}} = \bar{X}^2\mathfrak{g}_1,$$
$$\Theta_{C_{sg}} = \bar{X}\bar{Y}\left[\frac{\mathfrak{g}_1}{(N+1)}-\mathfrak{g}_{o1}\right],$$
$$\Theta_{D_{sg}} = \bar{Y}^2\left[1+\frac{3\mathfrak{g}_1}{8(N+1)^2}-\frac{\mathfrak{g}_{o1}}{2(N+1)}\right],$$
$$\Theta_{E_{sg}} = \bar{X}\bar{Y}\frac{\mathfrak{g}_1}{2(N+1)}.$$

Which is minimum for

$$\mathfrak{q}_{sg1}^{opt} = \left[\frac{\Theta_{B_{sg}}\Theta_{D_{sg}}-\Theta_{C_{sg}}\Theta_{E_{sg}}}{\Theta_{A_{sg}}\Theta_{B_{sg}}-\Theta^2_{C_{sg}}}\right],$$

and

$$\mathfrak{q}_{sg2}^{opt} = \left[\frac{\Theta_{A_{sg}}\Theta_{E_{sg}}-\Theta_{C_{sg}}\Theta_{D_{sg}}}{\Theta_{A_{sg}}\Theta_{B_{sg}}-\Theta^2_{C_{sg}}}\right].$$

The minimum MSE of $\hat{\mathfrak{T}}_{sg}$ is given by

$$\mathcal{MSE}_{min}(\hat{\mathfrak{T}}_{sg}) = \left[\bar{Y}^2-\frac{\Theta_{B_{sg}}\Theta^2_{D_{sg}}+\Theta_{A_{sg}}\Theta^2_{E_{sg}}-2\Theta_{C_{sg}}\Theta_{D_{sg}}\Theta_{E_{sg}}}{\Theta_{A_{sg}}\Theta_{B_{sg}}-\Theta^2_{C_{sg}}}\right]. \tag{2.23}$$

[9] introduced the generalized form of [22] estimator as follows

$$\hat{\mathfrak{T}}_{gk} = \left[\mathfrak{q}_{gk1}\bar{y}+\mathfrak{q}_{gk2}\left(\bar{X}-\bar{x}\right)\right] exp\left[\frac{\vartheta\left(\bar{X}-\bar{x}\right)}{\vartheta\left(\bar{X}+\bar{x}\right)+2\lambda}\right]. \tag{2.24}$$

For ease in calculation they use $\vartheta = 1$ and $\lambda = -1$.
The bias and MSE of $\hat{\mathfrak{T}}_{gk}$ are

$$\mathcal{B}(\hat{\mathfrak{T}}_{gk}) = \bar{Y}\left[(\mathfrak{q}_{gk1}-1)-\mathfrak{q}_{gk1}\theta_{gk}\mathfrak{g}_{o1}+\frac{3}{2}\mathfrak{q}_{gk1}\theta^2_{gk}\mathfrak{g}_1\right]-\mathfrak{q}_{gk2}\bar{X}\theta_{gk}\mathfrak{g}_1,$$

$$\mathcal{MSE}(\hat{\mathfrak{T}}_{gk}) = \bar{Y}^2+\mathfrak{q}^2_{gk1}\Theta_{A_{gk}}+\mathfrak{q}^2_{gk2}\Theta_{B_{gk}}+2\mathfrak{q}_{gk1}\mathfrak{q}_{gk2}\Theta_{C_{gk}}-2\mathfrak{q}_{gk1}\Theta_{D_{gk}}-2\mathfrak{q}_{gk2}\Theta_{E_{gk}}$$

where

$$\Theta_{A_{gk}} = \bar{Y}^2\left[1+\mathfrak{g}_o+4\theta^2_{gk}\mathfrak{g}_1-4\theta_{gk}\mathfrak{g}_{o1}\right],$$
$$\Theta_{B_{gk}} = \bar{X}^2\mathfrak{g}_1,$$
$$\Theta_{C_{gk}} = \bar{X}\bar{Y}\left[2\theta_{gk}\mathfrak{g}_1-\mathfrak{g}_{o1}\right],$$
$$\Theta_{D_{gk}} = \bar{Y}^2\left[1+\frac{3\theta^2_{gk}\mathfrak{g}_1}{2}-\theta_{gk}\mathfrak{g}_{o1}\right],$$
$$\Theta_{E_{gk}} = \bar{X}\bar{Y}\theta_{gk}\mathfrak{g}_1,$$
$$\theta_{gk} = \frac{\vartheta\bar{X}}{2\left(\vartheta\bar{X}+\lambda\right)}.$$

Which is minimum for

$$\mathfrak{q}_{gk1}^{opt} = \left[\frac{\Theta_{B_{gk}} \Theta_{D_{gk}} - \Theta_{C_{gk}} \Theta_{E_{gk}}}{\Theta_{A_{gk}} \Theta_{B_{gk}} - \Theta_{C_{gk}}^2} \right],$$

and

$$\mathfrak{q}_{gk2}^{opt} = \left[\frac{\Theta_{A_{gk}} \Theta_{E_{gk}} - \Theta_{C_{gk}} \Theta_{D_{gk}}}{\Theta_{A_{gk}} \Theta_{B_{gk}} - \Theta_{C_{gk}}^2} \right].$$

The minimum MSE of $\hat{\mathfrak{T}}_{gk}$ is

$$\mathcal{MSE}_{min}(\hat{\mathfrak{T}}_{gk}) = \left[\bar{Y}^2 - \frac{\Theta_{B_{gk}} \Theta_{D_{gk}}^2 + \Theta_{A_{gk}} \Theta_{E_{gk}}^2 - 2\Theta_{C_{gk}} \Theta_{D_{gk}} \Theta_{E_{gk}}}{\Theta_{A_{gk}} \Theta_{B_{gk}} - \Theta_{C_{gk}}^2} \right]. \tag{2.25}$$

[23] introduced the following difference-cum-exponential estimator for the estimation of population mean as follows

$$\hat{\mathfrak{T}}_j = \left[\frac{\bar{y}}{2} \left\{ exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right) + exp\left(\frac{\bar{x} - \bar{X}}{\bar{X} + \bar{x}} \right) \right\} + \mathfrak{q}_{j1}\bar{y} + \mathfrak{q}_{j2}\left(\bar{X} - \bar{x} \right) \right] exp\left[\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right]. \tag{2.26}$$

The bias and MSE of $\hat{\mathfrak{T}}_j$ are given by

$$\mathcal{B}(\hat{\mathfrak{T}}_j) = \bar{Y} \left[\mathfrak{q}_{j1}(1 - \frac{\mathfrak{g}_{o1}}{2} + \frac{3\mathfrak{g}_1}{8}) - \frac{\mathfrak{g}_{o1}}{2} \right] + \frac{1}{2}\mathfrak{q}_{j2}\left[\bar{X}\mathfrak{g}_1 + \bar{Y}\mathfrak{q}_{j2} \right],$$

$$\mathcal{MSE}(\hat{\mathfrak{T}}_j) = \bar{Y}^2 + \mathfrak{q}_{j1}^2 \Theta_{A_j} + \mathfrak{q}_{j2}^2 \Theta_{B_j} + 2\mathfrak{q}_{j1}\mathfrak{q}_{j2}\Theta_{C_j} - 2\mathfrak{q}_{j1}\Theta_{D_j} - 2\mathfrak{q}_{j2}\Theta_{E_j},$$

where

$$\begin{aligned}
\Theta_{L_j} &= \bar{Y}^2 \left[\mathfrak{g}_o - \mathfrak{g}_{o1} + \frac{1}{4}\mathfrak{g}_1 \right], \\
\Theta_{A_j} &= \bar{Y}^2 \left[1 + \mathfrak{g}_o + \mathfrak{g}_1 - 2\mathfrak{g}_{o1} \right], \\
\Theta_{B_j} &= \bar{X}^2 \mathfrak{g}_1, \\
\Theta_{C_j} &= \bar{X}\bar{Y} \left[\mathfrak{g}_1 - \mathfrak{g}_{o1} \right], \\
\Theta_{D_j} &= \bar{Y}^2 \left[\frac{3}{2}\mathfrak{g}_{o1} - \mathfrak{g}_o - \frac{3}{4}\mathfrak{g}_1 \right], \\
\Theta_{E_j} &= \bar{X}\bar{Y} \left[\mathfrak{g}_{o1} - \frac{1}{2}\mathfrak{g}_1 \right].
\end{aligned}$$

Which is minimum for

$$\mathfrak{q}_{j1}^{opt} = \left[\frac{\Theta_{B_j} \Theta_{D_j} - \Theta_{C_j} \Theta_{E_j}}{\Theta_{A_j} \Theta_{B_j} - \Theta_{C_j}^2} \right],$$

and

$$\mathfrak{q}_{j2}^{opt} = \left[\frac{\Theta_{A_j} \Theta_{E_j} - \Theta_{C_j} \Theta_{D_j}}{\Theta_{A_j} \Theta_{B_j} - \Theta_{C_j}^2} \right].$$

The minimum MSE of $\hat{\mathfrak{T}}_j$ is

$$\mathcal{MSE}_{min}(\hat{\mathfrak{T}}_j) = \left[\Theta_{L_j} - \frac{\Theta_{B_j} \Theta_{D_j}^2 + \Theta_{A_j} \Theta_{E_j}^2 - 2\Theta_{C_j} \Theta_{D_j} \Theta_{E_j}}{\Theta_{A_j} \Theta_{B_j} - \Theta_{C_j}^2} \right]. \tag{2.27}$$

[15] family of estimators is given by

$$\hat{\mathfrak{T}}_{kh} = \left[\mathfrak{q}_{kh1}\bar{y} + \mathfrak{q}_{kh2}\left(\bar{X} - \bar{x}\right)\right] exp\left[\frac{\bar{X} - \bar{x}}{2c_k\bar{X} - \bar{X} + \bar{x}}\right]. \tag{2.28}$$

The bias and MSE of $\hat{\mathfrak{T}}_{kh}$ are given by

$$\mathcal{B}(\hat{\mathfrak{T}}_{kh}) = \bar{Y}\left[(\mathfrak{q}_{kh1} - 1) + \mathfrak{q}_{kh1}\left\{\frac{3\mathfrak{g}_1}{8c_k^2} - \frac{\mathfrak{g}_{o1}}{2c_k}\right\}\right] + \mathfrak{q}_{kh2}\bar{X}\frac{\mathfrak{g}_1}{2c_k^2},$$

$$\mathcal{MSE}(\hat{\mathfrak{T}}_{kh}) = \bar{Y}^2 + \mathfrak{q}_{kh1}^2\Theta_{A_{kh}} + \mathfrak{q}_{kh2}^2\Theta_{B_{kh}} + 2\mathfrak{q}_{kh1}\mathfrak{q}_{kh2}\Theta_{C_{kh}} - 2\mathfrak{q}_{kh1}\Theta_{D_{kh}} - 2\mathfrak{q}_{kh2}\Theta_{E_{kh}},$$

where

$$\begin{aligned}
\Theta_{A_{kh}} &= \bar{Y}^2\left[1 + \mathfrak{g}_o + \frac{\mathfrak{g}_1}{c_k^2} - \frac{2\mathfrak{g}_{o1}}{c_k}\right],\\
\Theta_{B_{kh}} &= \bar{X}^2\mathfrak{g}_1,\\
\Theta_{C_{kh}} &= \bar{X}\bar{Y}\left[\frac{\mathfrak{g}_1}{c_k} - \mathfrak{g}_{o1}\right],\\
\Theta_{D_{kh}} &= \bar{Y}^2\left[1 + \frac{3\mathfrak{g}_1}{8c_k^2} - \frac{\mathfrak{g}_{o1}}{2c_k}\right],\\
\Theta_{E_{kh}} &= \bar{X}\bar{Y}\frac{\mathfrak{g}_1}{2c_k}.
\end{aligned}$$

Which is minimum for

$$\mathfrak{q}_{kh1}^{opt} = \left[\frac{\Theta_{B_{kh}}\Theta_{D_{kh}} - \Theta_{C_{kh}}\Theta_{E_{kh}}}{\Theta_{A_{kh}}\Theta_{B_{kh}} - \Theta_{C_{kh}}^2}\right],$$

and

$$\mathfrak{q}_{kh2}^{opt} = \left[\frac{\Theta_{A_{kh}}\Theta_{E_{kh}} - \Theta_{C_{kh}}\Theta_{D_{kh}}}{\Theta_{A_{kh}}\Theta_{B_{kh}} - \Theta_{C_{kh}}^2}\right].$$

The minimum MSE of $\hat{\mathfrak{T}}_{kh}$ is given by

$$\mathcal{MSE}_{min}(\hat{\mathfrak{T}}_{kh}) = \left[\bar{Y}^2 - \frac{\Theta_{B_{kh}}\Theta_{D_{kh}}^2 + \Theta_{A_{kh}}\Theta_{E_{kh}}^2 - 2\Theta_{C_{kh}}\Theta_{D_{kh}}\Theta_{E_{kh}}}{\Theta_{A_{kh}}\Theta_{B_{kh}} - \Theta_{C_{kh}}^2}\right]. \tag{2.29}$$

Obviously, a number of reviewed estimators constructed under a very specific modification/transformation. That's why, these estimators have restricted applications. So, we get here a motivation to define the very general class of estimators utilizing available auxiliary information in the next section.

3. PROPOSED GENERALIZED FAMILY OF ESTIMATORS

We propose the following generalized family of estimators as follows

$$\hat{\mathfrak{T}}_N = \left[w_1\bar{y}\left\{\frac{\left(a\bar{X} + b\right)}{\alpha\left(a\bar{x} + b\right) + (1 - \alpha)\left(a\bar{X} + b\right)}\right\}^g + w_2\left(\bar{X} - \bar{x}\right)\right] exp\left[\frac{\eta\left(\bar{X} - \bar{x}\right)}{\eta\left(2\xi\bar{X} - \phi\bar{X} + \bar{x}\right) + 2\lambda}\right]$$

where $\alpha, a, b, g, w_1, w_2, \eta, \phi, \xi$ and λ are all constants. When we write the above estimator with e terms under first order of approximation we get

Table 3: Family members of Khan et al. (2015)

Est.	c_k
$\hat{\mathfrak{T}}_{kh1} = \left[\mathfrak{q}_{kh1}\bar{y} + \mathfrak{q}_{kh2}\left(\bar{X}-\bar{x}\right)\right] exp\left[\dfrac{X-\bar{x}}{2C_x\bar{X}-\bar{X}+\bar{x}}\right]$	C_x
$\hat{\mathfrak{T}}_{kh2} = \left[\mathfrak{q}_{kh1}\bar{y} + \mathfrak{q}_{kh2}\left(\bar{X}-\bar{x}\right)\right] exp\left[\dfrac{\bar{X}-\bar{x}}{2\left(\rho+1\right)\bar{X}-\bar{X}+\bar{x}}\right]$	$\rho+1$
$\hat{\mathfrak{T}}_{kh3} = \left[\mathfrak{q}_{kh1}\bar{y} + \mathfrak{q}_{kh2}\left(\bar{X}-\bar{x}\right)\right] exp\left[\dfrac{\bar{X}-\bar{x}}{2\left(\dfrac{\rho+1}{2}\right)\bar{X}-\bar{X}+\bar{x}}\right]$	$\dfrac{\rho+1}{2}$
$\hat{\mathfrak{T}}_{kh4} = \left[\mathfrak{q}_{kh1}\bar{y} + \mathfrak{q}_{kh2}\left(\bar{X}-\bar{x}\right)\right] exp\left[\dfrac{\bar{X}-\bar{x}}{2\left(\dfrac{\rho+1}{3}\right)\bar{X}-\bar{X}+\bar{x}}\right]$	$\dfrac{\rho+1}{3}$

$$\hat{\mathfrak{T}}_N = \left[w_1\bar{Y}\left(1+e_o-\alpha\mathfrak{v}ge_1-\alpha\mathfrak{v}ge_oe_1+\frac{g(g+1)}{2}\alpha^2\mathfrak{v}^2e_1^2\right)-w_2\bar{X}e_1\right]\left[1-\gamma e_1+\frac{3}{2}\gamma^2e_1^2\right].$$

where

$$\gamma = \frac{\eta\bar{X}}{(b_k+1)\eta\bar{X}+2\lambda},\ b_k = 2\xi-\phi,\ k_1 = \alpha\mathfrak{v}+\gamma,\ k_2 = \left(\frac{g(g+1)}{2}\alpha^2\mathfrak{v}^2+\alpha\gamma\mathfrak{v}g+\frac{3}{2}\gamma^2\right).$$

Arranging the above equation and subtracting $\bar{Y}$ we have

$$\hat{\mathfrak{T}}_N-\bar{Y} = w_1\bar{Y}\left\{1+e_o-k_1e_1-k_1e_oe_1+k_2e_1^2\right\}-w_2\bar{X}\left\{e_1-\gamma e_1^2\right\}-\bar{Y}.$$

The bias of $\hat{\mathfrak{T}}_N$ is given by

$$B(\hat{\mathfrak{T}}_N) = w_1\bar{Y}\left\{1-k_1\mathsf{u}_{11}+\left(\frac{g(g+1)}{2}\alpha^2\mathfrak{v}^2+\alpha\gamma\mathfrak{v}g+\frac{3}{2}\gamma^2\right)\mathsf{u}_{02}\right\}+w_2\bar{X}\gamma\mathsf{u}_{02}-\bar{Y}.$$

The MSE of $\hat{\mathfrak{T}}_N$ is

$$\mathcal{MSE}(\hat{\mathfrak{T}}_N) = \bar{Y}^2+w_1^2\Delta_{A1}+w_2^2\Delta_{B1}+w_1w_2\Delta_{C1}+w_1\Delta_{D1}+w_2\Delta_{E1}$$

where

$$\Delta_{A1} = \bar{Y}^2\left\{1+\mathfrak{g}_o+\left(k_1^2+2k_2\right)\mathfrak{g}_1-4k_1\mathfrak{g}_{o1}\right\},$$
$$\Delta_{B1} = \bar{X}^2\mathfrak{g}_1,$$
$$\Delta_{C1} = 2\bar{X}\bar{Y}\left\{(k_1+\gamma)\mathfrak{g}_1-\mathfrak{g}_{o1}\right\},$$
$$\Delta_{D1} = 2\bar{Y}^2\left\{k_1\mathfrak{g}_{o1}-k_2\mathfrak{g}_1-1\right\},$$
$$\Delta_{E1} = -2\bar{X}\bar{Y}\gamma\mathfrak{g}_1.$$

By minimizing MSE of $\hat{\mathfrak{T}}_N$, we get the optimum values of w_1, w_2 as follows

$$w_1^{opt} = \left[\frac{-2\Delta_{B1}\Delta_{D1}+\Delta_{C1}\Delta_{E1}}{4\Delta_{A1}\Delta_{B1}-\Delta_{C1}^2}\right],$$

and

$$w_2^{opt} = \left[\frac{\Delta_{C1}\Delta_{D1} - 2\Delta_{A1}\Delta_{E1}}{4\Delta_{A1}\Delta_{B1} - \Delta_{C1}^2}\right].$$

Hence, the minimum mean square error of $\hat{\mathfrak{T}}_N$ is given by

$$\mathcal{MSE}_{min}(\hat{\mathfrak{T}}_N) = \left[\bar{Y}^2 - \frac{\Delta_{B1}\Delta_{D1}^2 + \Delta_{A1}\Delta_{E1}^2 - \Delta_{C1}\Delta_{D1}\Delta_{E1}}{4\Delta_{A1}\Delta_{B1} - \Delta_{C1}^2}\right]. \tag{3.1}$$

3.1. Some special cases of $\hat{\mathfrak{T}}_N$

Note that a number of reviewed estimators or reviewed family of estimators are the members of proposed generalized family, Such as

- when $(a, b, \alpha, g, \eta, \xi, \phi, \lambda, w_1, w_2) = (-, -, -, 0, 0, -, -, 1, 1, 0)$,
  Then $\hat{\mathfrak{T}}_N$ converted into usual unbiased estimator.
- when $(a, b, \alpha, g, \eta, \xi, \phi, \lambda, w_1, w_2) = (-, -, -, 0, 0, -, -, 1, w_1, 0)$,
  Then $\hat{\mathfrak{T}}_N$ converted into [21] estimator.
- when $(a, b, \alpha, g, \eta, \xi, \phi, \lambda, w_1, w_2) = (-, -, -, 0, 1, 1, 1, 0, 1, 0)$,
  Then $\hat{\mathfrak{T}}_N$ converted into [5] estimator.
- when $(a, b, \alpha, g, \eta, \xi, \phi, \lambda, w_1, w_2) = (-, -, -, 0, 0, -, -, 1, w_1, w_2)$,
  Then $\hat{\mathfrak{T}}_N$ converted into [20] estimator.
- when $(a, b, \alpha, g, \eta, \xi, \phi, \lambda, w_1, w_2) = (a, b, \alpha, g, 0, -, -, 1, 1, 0)$,
  Then $\hat{\mathfrak{T}}_N$ converted into [16] estimator.
- when $(a, b, \alpha, g, \eta, \xi, \phi, \lambda, w_1, w_2) = (a, b, \alpha, g, 0, -, -, 1, w_1, 0)$,
  Then $\hat{\mathfrak{T}}_N$ converted into [17] estimator.
- when $(a, b, \alpha, g, \eta, \xi, \phi, \lambda, w_1, w_2) = (-, -, -, 0, \eta, 1, 1, \lambda, 1, 0)$,
  Then $\hat{\mathfrak{T}}_N$ converted into [26] family of estimators.
- when $(a, b, \alpha, g, \eta, \xi, \phi, \lambda, w_1, w_2) = (-, -, -, 0, 1, 1, 1, 0, w_1, w_2)$,,
  Then $\hat{\mathfrak{T}}_N$ converted into [8] estimator.
- when $(a, b, \alpha, g, \eta, \xi, \phi, \lambda, w_1, w_2) = (-, -, -, 0, 1, N, -1, 0, w_1, w_2)$,,
  Then $\hat{\mathfrak{T}}_N$ converted into [22] estimator.
- when $(a, b, \alpha, g, \eta, \xi, \phi, \lambda, w_1, w_2) = (-, -, -, 0, \eta, 1, 1, \lambda, w_1, 0)$,
  Then $\hat{\mathfrak{T}}_N$ converted into Yadav and Kadilar (2013) family of estimators estimator.
- when $(a, b, \alpha, g, \eta, \xi, \phi, \lambda, w_1, w_2) = (-, -, -, 0, \eta, 1, 1, \lambda, w_1, w_2)$,
  Then $\hat{\mathfrak{T}}_N$ converted into [9] estimator.
- when $(a, b, \alpha, g, \eta, \xi, \phi, \lambda, w_1, w_2) = (-, -, -, 0, 1, \xi, 1, 0, w_1, w_2)$,
  Then $\hat{\mathfrak{T}}_N$ converted into [15] family of estimators.

Table 4: Some new members of proposed family of estimators

Est.	a	b	α	g	η	ξ	ϕ	λ
$\hat{\mathfrak{T}}_{N1}$	1	0	1	1	1	ρ	1	-1
$\hat{\mathfrak{T}}_{N2}$	1	0	1	1	1	C_x	1	-1
$\hat{\mathfrak{T}}_{N3}$	1	0	1	1	1	$\beta_2(x)$	1	-1
$\hat{\mathfrak{T}}_{N4}$	1	0	1	1	1	$\beta_1(x)$	1	-1
$\hat{\mathfrak{T}}_{N5}$	1	0	1	1	1	N	1	-1
$\hat{\mathfrak{T}}_{N6}$	1	0	1	1	1	1	1	-1
$\hat{\mathfrak{T}}_{N7}$	1	0	1	1	1	$1-\frac{n}{N}$	1	-1

4. EFFICIENCY COMPARISON

Here we perform theoretical efficiency comparison for the proposed estimators by looking at the MSE of the reviewed estimators as given below

Observation (1):

$$\mathcal{MSE}_{min}(\hat{\mathfrak{T}}_N) < \mathcal{MSE}(\hat{\mathfrak{T}}_r),$$

if

$$\left[\frac{\Delta_{B1}\Delta_{D1}^2+\Delta_{A1}\Delta_{E1}^2-\Delta_{C1}\Delta_{D1}\Delta_{E1}}{4\Delta_{A1}\Delta_{B1}-\Delta_{C1}^2}\right]-\bar{Y}^2\left[1-(\mathfrak{g}_o+\mathfrak{g}_1-2\mathfrak{g}_{o1})\right]>0$$

Observation (2):

$$\mathcal{MSE}_{min}(\hat{\mathfrak{T}}_N) < \mathcal{MSE}(\hat{\mathfrak{T}}_{reg}),$$

if

$$\left[\frac{\Delta_{B1}\Delta_{D1}^2+\Delta_{A1}\Delta_{E1}^2-\Delta_{C1}\Delta_{D1}\Delta_{E1}}{4\Delta_{A1}\Delta_{B1}-\Delta_{C1}^2}\right]-\bar{Y}^2\left[1-\left(\mathfrak{g}_o-\frac{\mathfrak{g}_{01}^2}{\mathfrak{g}_1}\right)\right]>0$$

Observation (3):

$$\mathcal{MSE}_{min}(\hat{\mathfrak{T}}_N) < \mathcal{MSE}(\hat{\mathfrak{T}}_{bt}),$$

if

$$\left[\frac{\Delta_{B1}\Delta_{D1}^2+\Delta_{A1}\Delta_{E1}^2-\Delta_{C1}\Delta_{D1}\Delta_{E1}}{4\Delta_{A1}\Delta_{B1}-\Delta_{C1}^2}\right]-\bar{Y}^2\left[1-(\mathfrak{g}_o+\frac{1}{4}\mathfrak{g}_1-\mathfrak{g}_{o1})\right]>0$$

Observation (4):

$$\mathcal{MSE}_{min}(\hat{\mathfrak{T}}_N) < \mathcal{MSE}(\hat{\mathfrak{T}}_{rao}),$$

if

$$\left[\frac{\Delta_{B1}\Delta_{D1}^2+\Delta_{A1}\Delta_{E1}^2-\Delta_{C1}\Delta_{D1}\Delta_{E1}}{4\Delta_{A1}\Delta_{B1}-\Delta_{C1}^2}\right]-\left[\frac{\Theta_{B_{rao}}\Theta_{D_{rao}}^2}{\Theta_{A_{rao}}\Theta_{B_{rao}}-\Theta_{C_{rao}}^2}\right]>0$$

Observation (5):

$$\mathcal{MSE}_{min}(\hat{\mathfrak{T}}_N) < \mathcal{MSE}(\hat{\mathfrak{T}}_{kk}),$$

if

$$\left[\frac{\Delta_{B1}\Delta_{D1}^2+\Delta_{A1}\Delta_{E1}^2-\Delta_{C1}\Delta_{D1}\Delta_{E1}}{4\Delta_{A1}\Delta_{B1}-\Delta_{C1}^2}\right]-\left[\frac{\Theta_{A_{kk}^2}}{\Theta_{B_{kk}}}\right]>0$$

Observation (6):

$$\mathcal{MSE}_{min}(\hat{\mathfrak{T}}_N) < \mathcal{MSE}(\hat{\mathfrak{T}}_{lp}),$$

if

$$\left[\frac{\Delta_{B1}\Delta_{D1}^2 + \Delta_{A1}\Delta_{E1}^2 - \Delta_{C1}\Delta_{D1}\Delta_{E1}}{4\Delta_{A1}\Delta_{B1} - \Delta_{C1}^2}\right] - \left[\frac{\Theta_{B_{lp}}\Theta_{D_{lp}}^2 + \Theta_{A_{lp}}\Theta_{E_{lp}}^2 - 2\Theta_{C_{lp}}\Theta_{D_{lp}}\Theta_{E_{lp}}}{\Theta_{A_{lp}}\Theta_{B_{lp}} - \Theta_{C_{lp}}^2}\right] > 0$$

Observation (7):

$$\mathcal{MSE}_{min}(\hat{\mathfrak{T}}_N) < \mathcal{MSE}(\hat{\mathfrak{T}}_{sg}),$$

if

$$\left[\frac{\Delta_{B1}\Delta_{D1}^2 + \Delta_{A1}\Delta_{E1}^2 - \Delta_{C1}\Delta_{D1}\Delta_{E1}}{4\Delta_{A1}\Delta_{B1} - \Delta_{C1}^2}\right] - \left[\frac{\Theta_{B_{sg}}\Theta_{D_{sg}}^2 + \Theta_{A_{sg}}\Theta_{E_{sg}}^2 - 2\Theta_{C_{sg}}\Theta_{D_{sg}}\Theta_{E_{sg}}}{\Theta_{A_{sg}}\Theta_{B_{sg}} - \Theta_{C_{sg}}^2}\right] > 0$$

Observation (8):

$$\mathcal{MSE}_{min}(\hat{\mathfrak{T}}_N) < \mathcal{MSE}(\hat{\mathfrak{T}}_{yk}),$$

if

$$\left[\frac{\Delta_{B1}\Delta_{D1}^2 + \Delta_{A1}\Delta_{E1}^2 - \Delta_{C1}\Delta_{D1}\Delta_{E1}}{4\Delta_{A1}\Delta_{B1} - \Delta_{C1}^2}\right] - \left[\frac{\Theta_{A_{yk}^2}}{\Theta_{B_{yk}}}\right] > 0$$

Observation (9):

$$\mathcal{MSE}_{min}(\hat{\mathfrak{T}}_N) < \mathcal{MSE}(\hat{\mathfrak{T}}_{gk}),$$

if

$$\left[\frac{\Delta_{B1}\Delta_{D1}^2 + \Delta_{A1}\Delta_{E1}^2 - \Delta_{C1}\Delta_{D1}\Delta_{E1}}{4\Delta_{A1}\Delta_{B1} - \Delta_{C1}^2}\right] - \left[\frac{\Theta_{B_{gk}}\Theta_{D_{gk}}^2 + \Theta_{A_{gk}}\Theta_{E_{gk}}^2 - 2\Theta_{C_{gk}}\Theta_{D_{gk}}\Theta_{E_{gk}}}{\Theta_{A_{gk}}\Theta_{B_{gk}} - \Theta_{C_{gk}}^2}\right] > 0$$

Observation (10):

$$\mathcal{MSE}_{min}(\hat{\mathfrak{T}}_N) < \mathcal{MSE}(\hat{\mathfrak{T}}_j),$$

if

$$\left\{\left[\frac{\Delta_{B1}\Delta_{D1}^2 + \Delta_{A1}\Delta_{E1}^2 - \Delta_{C1}\Delta_{D1}\Delta_{E1}}{4\Delta_{A1}\Delta_{B1} - \Delta_{C1}^2}\right] - \left[\frac{\Theta_{B_j}\Theta_{D_j}^2 + \Theta_{A_j}\Theta_{E_j}^2 - 2\Theta_{C_j}\Theta_{D_j}\Theta_{E_j}}{\Theta_{A_j}\Theta_{B_j} - \Theta_{C_j}^2}\right]\right\} - [\Theta_{L_j} - \bar{Y}^2] > 0$$

Observation (11):

$$\mathcal{MSE}_{min}(\hat{\mathfrak{T}}_N) < \mathcal{MSE}(\hat{\mathfrak{T}}_{kh}),$$

if

$$\left[\frac{\Delta_{B1}\Delta_{D1}^2 + \Delta_{A1}\Delta_{E1}^2 - \Delta_{C1}\Delta_{D1}\Delta_{E1}}{4\Delta_{A1}\Delta_{B1} - \Delta_{C1}^2}\right] - \left[\frac{\Theta_{B_{kh}}\Theta_{D_{kh}}^2 + \Theta_{A_{kh}}\Theta_{E_{kh}}^2 - 2\Theta_{C_{kh}}\Theta_{D_{kh}}\Theta_{E_{kh}}}{\Theta_{A_{kh}}\Theta_{B_{kh}} - \Theta_{C_{kh}}^2}\right] > 0$$

From above observations we can argue that the new estimators perform better than all of the reviewed estimators.

5. NUMERICAL ILLUSTRATION

For assessing the performance of proposed and reviewed estimators, we make use the data set available in [18] concerning Output for 80 factories in a region considered as (Y) and Fixed Capital as (X). Details of the population descriptives are, $n = 25$, $\bar{Y} = 5182.637$, $S_y = 1835.659$, $\bar{X} = 1126.463$, $S_x = 845.6097$, $\rho = 0.941$ and $N = 80$. The MSEs and biases of all reviewed and proposed estimators are calculated and summarized in Table 5. When we examine the Table 5, we can say that members of our generalized family of estimators have minimum MSEs compared to existing estimators. We can also see that our proposed estimators are more efficient than regression estimator. It is well known that regression estimator and classical mean estimators are unbiased. However the ratio and exponential type estimators are biased. From Table 5, we can conclude that members of our proposed estimators have smaller biases.

Table 5: MSEs and biases of proposed and existing estimators

Est.	$\|Bias\|$	MSE	Est.	$\|Bias\|$	MSE	Est.	$\|Bias\|$	MSE
$\hat{\mathfrak{T}}_o$	-	92665.16	$\hat{\mathfrak{T}}_{kk6}$	13.48	17195.69	$\hat{\mathfrak{T}}_{kh2}$	79.23	10536.54
$\hat{\mathfrak{T}}_r$	57.84	139168.40	$\hat{\mathfrak{T}}_{kk7}$	25.99	18796.51	$\hat{\mathfrak{T}}_{kh3}$	26.73	10397.59
$\hat{\mathfrak{T}}_{reg}$	-	10558.63	$\hat{\mathfrak{T}}_{kk8}$	66.20	59920.87	$\hat{\mathfrak{T}}_{kh4}$	156.61	9882.26
$\hat{\mathfrak{T}}_{bt}$	15.23	11857.50	$\hat{\mathfrak{T}}_{kk9}$	142.49	133807.40	$\hat{\mathfrak{T}}_{N1}$	16.50	7249.20
$\hat{\mathfrak{T}}_{rao}$	19.01	26859.01	$\hat{\mathfrak{T}}_{kk10}$	141.76	133071.70	$\hat{\mathfrak{T}}_{N2}$	10.58	6465.57
$\hat{\mathfrak{T}}_{kk1}$	24.26	92346.57	$\hat{\mathfrak{T}}_{lp}$	20.03	10412.84	$\hat{\mathfrak{T}}_{N3}$	20.01	8303.64
$\hat{\mathfrak{T}}_{kk2}$	142.85	134168.70	$\hat{\mathfrak{T}}_{sg}$	107.54	10554.47	$\hat{\mathfrak{T}}_{N4}$	19.75	7519.92
$\hat{\mathfrak{T}}_{kk3}$	142.56	133880.50	$\hat{\mathfrak{T}}_{gk}$	54.60	8766.89	$\hat{\mathfrak{T}}_{N5}$	12.77	8417.13
$\hat{\mathfrak{T}}_{kk4}$	142.75	134068.10	$\hat{\mathfrak{T}}_j$	19.36	10069.72	$\hat{\mathfrak{T}}_{N6}$	18.14	7395.20
$\hat{\mathfrak{T}}_{kk5}$	141.40	132709.60	$\hat{\mathfrak{T}}_{kh1}$	86.24	10164.66	$\hat{\mathfrak{T}}_{N7}$	10.56	6017.95

6. CONCLUSION

In this article we propose an improved generalized family of estimators which covers a number of estimators from literature. Also many more estimators can be generated in future through from proposed generalized family. Theoretical properties such as bias and MSE of the proposed generalized family are determined under first order of approximation. Moreover, numerical illustration reveals that the proposed estimators are better than existing estimators in terms of MSE. So, we recommend the utilization of the proposed estimator for getting more efficient estimation of $\bar{Y}$ under simple random sampling.

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