# AN EXTENSION OF POWER DISTRIBUTION

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**ABSTRACT**
This article discusses the derivation of different statistical measures associated to four parameter extension of power distribution. The extension of power distribution has been introduced by using the concept of transmutation and weighted distribution. This extension is named as weighted transmuted power distribution and abbreviated as WTPD. Various properties of WTPD have been studied in detail. Random numbers from the introduced extension are generated by using the Inverse sampling method. Parameters are estimated by using the maximum likelihood estimation (MLE) method. Comparison is made between special cases of WTPD in terms of fitting to different types of data sets. AIC (Akaike Information Criterion) and Kolmogorov-Smirnov test (one sample, two sided) is used as a tool for making the comparison. The motive behind the construction of WTPD and its fitting to some considered data sets, is to show its importance and support its validity in describing a random phenomena.

**KEYWORDS:** Quadratic rank transmutation map (Q.R.T.M.), Weighted Distribution, Entropy, Order Statistics, MLE, AIC, Kolmogorov-Smirnov test.

**MSC**: 60E05

**RESUMEN**
En este paper se discute la derivation de differentes medidas estadisticaa associatdas a la e etesion de Potencia de cuatro parametros. Las extensiones ahs sido inroducdas usnado el concepto de distribcion con transmutation poderacon weighted distribution. Esta extension es llamda distribion de potentica transmutada, abrevadmante WTPD. Varias propiedade de la WTPD han sido estudiedas en detalle. Numeros aleatorios de la introducida extension son generados usando el metodode muestreo Inverso. Son estimdors Pparameters ausando el metoo d ela estimcion Maximo Verosimil (MLE) Se hacen Comparaciones entre casos espciles de WTPD en terminos del ajsute de differentes conjuntos de tipos de data . AIC (Akaike Information Criterion) y el tst de Kolmogorov-Smirnov (de una muestra y dos colas ) ison usada como hermeitas para hacer las comparisones. La motivacion para la construction ode WTPD y su ajste de algunso conjutos d eatos considerados, es stblcer su importancia y sorpotar su vlaidez para desciribir fenomnso aleaorios.

**PALABRAS CLAVE :** Mapeo de la transmutacion de rankeo cadratico (Q.R.T.M.), Distribution poderada, Entropiay, estadísticos de Orden, MLE, AIC, test de Kolmogorov-Smirnov test.

## 1. INTRODUCTION

Summarizing irregular events in which chance plays a noteworthy part forms the fundamentals of Statistics. Being an analyst, it is imperative to predict and anticipate some future events with significantly higher precision and the level of exactness can be ensured only by using the reasonable probability models. In the present times, application of statistics, especially statistical modelling is vast to the point that there is merely any field where statistical insights are not given the vital importance. Statistical insights can be proved fruitful only if right kind of statistical models and techniques are used to reveal the hidden characteristics of a statistical population. Since, there is considerably a large family of probability models that can be used to express the characteristics of a random process. These models may not be enough to express all the real world random processes and the interest to develop generally a larger family of probability models emerge. Different techniques exist in statistical literature that are meant for developing and introducing the new extensions of an already existing probability distribution e.g. the concept of G-distribution, truncation, exponentiation, transmutation etc. One of the features of these newly introduce extensions is that they usually contain more parameters, while it is sufficient to use at most four-parameter extension for reasonable purposes, see [17]. The noticeable change on using the extensions containing more than four parameters is thought to be vague. Generally, different types of G-distributions are used to introduce a new version of an already existing distribution. There is a lot of well-known class of G-distributions that exist in literature, see

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for instance [4], [16], [21] and [2]. Different statistical properties of these newly introduced family of distributions are studied in detail, see for example [1], [2], [4], [12], [9] and [14] etc.

Every statistical investigation begins with the collection of data. Therefore, a data analyst must be careful enough to examine whether the observation gets selected and recorded with equal probabilities or not. So, it is imperative to investigate the sampling methods to be used for drawing a sample, before we go for factual findings about the population. Circumstances exist, under which an observation gets selected with relatively larger probability or with probability proportional to a function known as weight function e.g. in encountered sampling. Under such circumstances it would be fair to use the weighted versions of some already existing probability distribution. Weighted distributions take into consideration the method of ascertainment by adjusting the probabilities of actual occurrence of an event, neglecting such adjustments may lead a statistician to come up with wrong insights. The concept of weighted distributions finds its existance from Fisher [8], in which it is observed that how ascertainment can influence the distribution of recorded observations. Later, it was Rao [19] who generalised the findings of Fisher [8]. In [19], various situations were recognized that can be modeled by weighted distributions. Many newly introduced distributions along with their weighted versions exist in literature whose statistical behavior is extensively studied during recent decades, see for example [5], [6], [3], [11], [13] and [15].

In this paper, we have investigated different statistical properties of a four parameter extension of power distribution. On assuming the probability of drawing an observation proportional to weight function $x^{\omega}$ and using quadratic rank transmutation map (QRTM) we derived the density of WTPD in Section 2.

## 2 . DERIVATION OF WTPD

A random variable $X$ is said to follow power distribution if it's probability density function (*p.d.f.*) and cumulative distribution function (*c.d.f.*) are respectively given by (2..1) and (2..2).

$$f(x;\alpha,\theta)=\frac{\alpha x^{\alpha-1}}{\theta^{\alpha}},0\le x\le\theta,\theta>0,\alpha>0 \qquad (2..1)$$

$$F(x;\alpha,\theta)=\left(\frac{x}{\theta}\right)^{\alpha}. \qquad (2..2)$$

QRTM, which is the function of a base line *c.d.f.* (i.e. $F(x)$) and transmutation parameter $\beta$, was introduced in [20] and is given by (2..3).

$$F_t(x)=(1+\beta)F(x)-\beta\{F(x)\}^2,\ |\beta|\le 1 \qquad (2..3)$$

where, $F_t(x)$ is the transmuted *c.d.f.*

Now, on using (2..2) in (2..3) as the base line *c.d.f.*, we get the *c.d.f.* of transmuted power function distribution as given by (2..4).

$$F_t(x;\alpha,\theta,\beta)=\frac{(1+\beta)(\theta x)^{\alpha}-\beta x^{2\alpha}}{\theta^{2\alpha}},\ |\,\beta\,|\le 1. \qquad (2..4)$$

Differentiating equation (2..4) w.r.t. $x$, we obtain the *p.d.f.* of transmuted power distribution and is given by (2..5).

$$f_t(x;\alpha,\theta,\beta)=\frac{\alpha\theta^{\alpha}(1+\beta)x^{\alpha-1}-2\beta\alpha x^{2\alpha-1}}{\theta^{2\alpha}},\ 0\le x\le\theta,\theta>0,\alpha>0,|\beta|\le 1. \qquad (2..5)$$

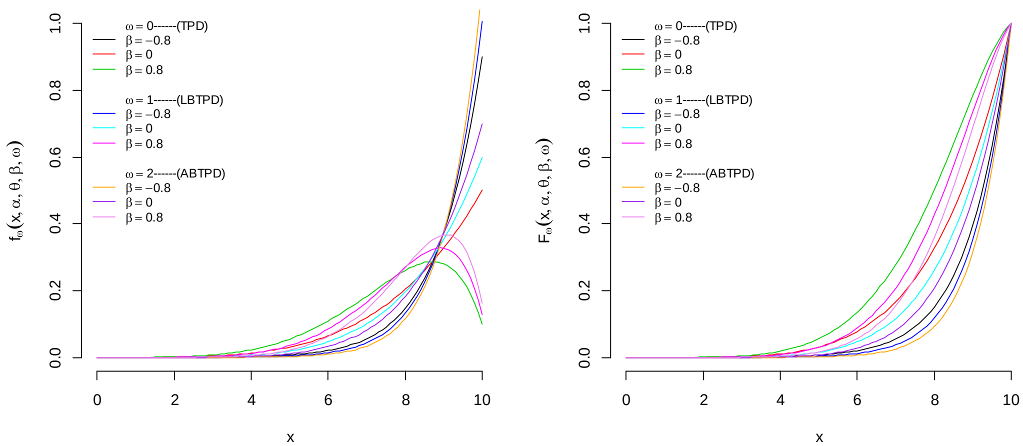
Considering the weight function $w(x)=x^{\omega}$, and using the definition of weighted distribution, the *p.d.f.* of WTPD is given by:

$$f_{\omega}(x;\alpha,\theta,\beta,\omega)=\frac{x^{\omega}f_t(x)}{E[x^{\omega}]}$$ where, $\omega>0$ is the weight parameter.

$$f_\omega(x;\alpha,\theta,\beta,\omega) = \frac{(\alpha+\omega)(2\alpha+\omega)x^{\alpha+\omega-1}\{\theta^{\alpha}(1+\beta)-2\beta x^{\alpha}\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)}, 0 \le x \le \theta, \theta > 0, \alpha > 0, \omega > 0, |\beta| \le 1. \quad (2..6)$$

Therefore, the *c.d.f.* of WTPD is given by (2..7).

$$F_\omega(x;\alpha,\theta,\beta,\omega) = \frac{x^{\alpha+\omega}\{\theta^{\alpha}(1+\beta)(2\alpha+\omega) - 2x^{\alpha}\beta(\alpha+\omega)\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)} \quad (2..7)$$




(a) Density plots (b) *c.d.f.* plots

Figure 1: Density and *c.d.f.* plots of WTPD at different values of $\theta, \alpha, \beta$ and $\omega$.

## 3 STRUCTURAL PROPERTIES OF WTPD

In this Section, we investigate various structural properties of WTPD

**Theorem 3.1.** If a random variable $X$ follows WTPD, then its $r^{th}$ moment about origin is given by:

$$\mu_r' = E[x^r] = \frac{\theta^r(\alpha+\omega)(2\alpha+\omega)\{2\alpha-(\beta-1)(\omega+r)\}}{(r+\alpha+\omega)(r+2\alpha+\omega)(2\alpha+\omega-\beta\omega)}, \; r = 1,2,3,... \quad (3..1)$$

**Proof.**

$$\mu_r' = \int_0^\theta x^r f_\omega(x;\alpha,\theta,\beta,\omega)\, dx$$

$$\mu_r' = \int_0^\theta x^r \frac{(\alpha+\omega)(2\alpha+\omega)x^{\alpha+\omega-1}\{\theta^{\alpha}(1+\beta)-2\beta x^{\alpha}\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)}\, dx$$

After simplifying the expression, we get:

$$\mu_r' = \frac{\theta^r(\alpha+\omega)(2\alpha+\omega)\{2\alpha-(\beta-1)(\omega+r)\}}{(r+\alpha+\omega)(r+2\alpha+\omega)(2\alpha+\omega-\beta\omega)}$$

Substituting $r=1$, we obtain the Mean of WTPD and is given by (3.2)

$$\mu_1' = \frac{\theta(\alpha+\omega)(2\alpha+\omega)\{2\alpha-(\beta-1)(\omega+1)\}}{(1+\alpha+\omega)(1+2\alpha+\omega)(2\alpha+\omega-\beta\omega)} \quad (3..2)$$

Similarly, the variance of WTPD is obtained by subtracting the square of mean from the $\mu_2'$ and is given by (3..3)

$$\sigma^2 = \frac{\theta^2(\alpha+\omega)(2\alpha+\omega)\{2\alpha-(\beta-1)(\omega+2)\}}{(2+\alpha+\omega)(2+2\alpha+\omega)(2\alpha+\omega-\beta\omega)} - \left[\frac{\theta(\alpha+\omega)(2\alpha+\omega)\{2\alpha-(\beta-1)(\omega+1)\}}{(1+\alpha+\omega)(1+2\alpha+\omega)(2\alpha+\omega-\beta\omega)}\right]^2 \qquad (3..3\cdot)$$

Higher order moments like skewness and kurtosis have been computed numerically at different values of $\theta, \alpha, \beta, \omega$ and are reported in Table-1.

**Theorem 3.2.** The moment generating function and characteristic function of a random variable $X$ following WTPD are respectively given by:

$$M_x(t) = \frac{(\alpha+\omega)(2\alpha+\omega)\left[\theta^\alpha(1+\beta)\sum_{j=0}^{\infty}\frac{t^j\theta^{\alpha+\omega+j}}{j!(\alpha+\omega+j)} - 2\beta\sum_{j=0}^{\infty}\frac{t^j\theta^{2\alpha+\omega+j}}{j!(2\alpha+\omega+j)}\right]}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)} \qquad (3..4)$$

$$\Phi_x(t) = \frac{(\alpha+\omega)(2\alpha+\omega)\left[\theta^\alpha(1+\beta)\sum_{j=0}^{\infty}\frac{(i\,t)^j\theta^{\alpha+\omega+j}}{j!(\alpha+\omega+j)} - 2\beta\sum_{j=0}^{\infty}\frac{(i\,t)^j\theta^{2\alpha+\omega+j}}{j!(2\alpha+\omega+j)}\right]}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)} \qquad (3..5)$$

**Proof.** From the definition of m.g.f. we have:

$$M_x(t) = E[e^{tx}]$$

$$M_x(t) = \int_0^\infty e^{tx} f_\omega(x;\alpha,\theta,\beta,\omega)dx$$

$$M_x(t) = \int_0^\theta e^{tx}\,\frac{(\alpha+\omega)(2\alpha+\omega)x^{\alpha+\omega-1}\{\theta^\alpha(1+\beta)-2\beta x^\alpha\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)}\,dx.$$

On simplifying the integral, we obtained.

$$M_x(t) = \frac{(\alpha+\omega)(2\alpha+\omega)}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)}\left[\theta^\alpha(1+\beta)\sum_{j=0}^{\infty}\frac{t^j\theta^{\alpha+\omega+j}}{j!(\alpha+\omega+j)} - 2\beta\sum_{j=0}^{\infty}\frac{t^j\theta^{2\alpha+\omega+j}}{j!(2\alpha+\omega+j)}\right]$$

Similarly, Characteristic function is be obtained by using the following relation:

$$\Phi_x(t) = M_x(it)$$

$$\Phi_x(t) = \frac{(\alpha+\omega)(2\alpha+\omega)}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)}\left[\theta^\alpha(1+\beta)\sum_{j=0}^{\infty}\frac{(it)^j\theta^{\alpha+\omega+j}}{j!(\alpha+\omega+j)} - 2\beta\sum_{j=0}^{\infty}\frac{(it)^j\theta^{2\alpha+\omega+j}}{j!(2\alpha+\omega+j)}\right]$$

### 3.1 Reliability Analysis:

The Reliabilty fuction and hazard rate associated with WTPD are respectively given by (3..6) and (3..7).

$$R_\omega(t) = 1 - \frac{t^{\alpha+\omega}\{\theta^\alpha(1+\beta)(2\alpha+\omega) - 2t^\alpha\beta(\alpha+\omega)\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)} \qquad (3..6)$$

$$h_\omega(t) = \frac{(\alpha+\omega)(2\alpha+\omega)\,x^{\alpha+\omega-1}\{\theta^\alpha(1+\beta)-2\beta\,x^\alpha\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega) - t^{\alpha+\omega}\{\theta^\alpha(1+\beta)(2\alpha+\omega)-2t^\alpha\beta(\alpha+\omega)\}} \qquad (3..7)$$

**Theorem 3.3.** Mean residual life (M.R.L.) function of a random variable $X$ following WTPD is given as:

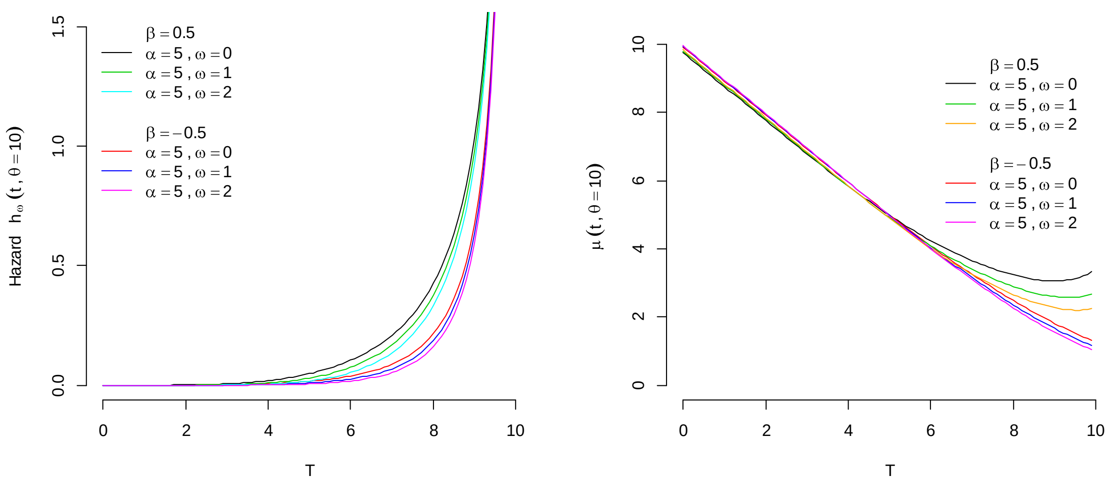
$$\mu(t) = \frac{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)(\theta-t)-\left[\frac{\theta^{\alpha}(1+\beta)(2\alpha+\omega)(\theta^{\alpha+\omega+1}-t^{\alpha+\omega+1})}{(\alpha+\omega+1)}-\frac{2\beta(\alpha+\omega)(\theta^{\alpha+1}-t^{\alpha+1})}{\alpha+1}\right]}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)-t^{\alpha+\omega}\{\theta^{\alpha}(1+\beta)(2\alpha+\omega)-2t^{\alpha}\beta(\alpha+\omega)\}} \quad (3..8)$$

**Proof.** By the definition of M.R.L., we have:

$$\mu(t) = \frac{\int_t^{\infty} R(x)dx}{R(t)} = \frac{\int_t^{\theta} R(x)dx}{R(t)}$$

$$\mu(t) = \frac{\int_t^{\theta}\left[1-\frac{x^{\alpha+\omega}\{\theta^{\alpha}(1+\beta)(2\alpha+\omega)-2x^{\alpha}\beta(\alpha+\omega)\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)}\right]dx}{1-\frac{t^{\alpha+\omega}\{\theta^{\alpha}(1+\beta)(2\alpha+\omega)-2t^{\alpha}\beta(\alpha+\omega)\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)}}$$

$$\mu(t) = \frac{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)(\theta-t)-\left[\frac{\theta^{\alpha}(1+\beta)(2\alpha+\omega)(\theta^{\alpha+\omega+1}-t^{\alpha+\omega+1})}{(\alpha+\omega+1)}-\frac{2\beta(\alpha+\omega)(\theta^{\alpha+1}-t^{\alpha+1})}{\alpha+1}\right]}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)-t^{\alpha+\omega}\{\theta^{\alpha}(1+\beta)(2\alpha+\omega)-2t^{\alpha}\beta(\alpha+\omega)\}}$$




(a) Hazard curve (b) M.R.L. curve

Figure 2: Hazard rate and Mean residual life function at different values of $\theta, \alpha, \beta$ and $\omega$.

From Fig. 2a, it is quite clear that WTPD posses increasing hazard rate which means it has decreasing mean residual life (D.M.R.L.), that is evident from Fig. 2b. Since, D.M.R.L. implies N.B.U.E. (new better than used in expectation i.e. $\mu(0) \geq \mu(t), \forall t \geq 0$ ), therefore WTPD belongs to N.B.U.E. family. WTPD belonging to N.B.U.E. family is also justified as per the result of Hall and Wellner [10] which states that for N.B.U.E. family, coefficient of variation has to be $\leq 1$. From Table-2, it is quite evident that $c.v.$ is $\leq 1$ for WTPD.

### 3.2 Entropy Measure

**Theorem 3.4.** Renyi entropy of order $\delta$ of WTPD is given by:

$$H_R(\delta) = \frac{1}{1-\delta}\log\left[\left\{\frac{(\alpha+\omega)(2\alpha+\omega)}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)}\right\}^{\delta}\sum_{k=0}^{\infty}\binom{\delta}{k}\{\theta^{\alpha}(1+\beta)\}^{\delta-k}(-2\beta)^{k}\frac{\theta^{\delta(\alpha+\omega-1)+\alpha k+1}}{\delta(\alpha+\omega-1)+\alpha k+1}\right] \quad (3..9)$$

**Proof.** Entropy measure proposed by Alferd Renyi known as Renyi entropy of order $\delta$ for a random variable $X$ is given by:

$$H_R(\delta) = \frac{1}{1-\delta} \log\left[\int_0^{\infty} \{f(x)\}^{\delta} dx\right]; \delta \geq 0 \,\&\, \delta \neq 1 \quad (3..10)$$

Therefore, from (3..10), Renyi entropy for WTPD is given as:

$$H_R(\delta) = \frac{1}{1-\delta} \log\left[\int_0^{\theta} \left\{ \frac{(\alpha+\omega)(2\alpha+\omega)x^{\alpha+\omega-1}\{\theta^{\alpha}(1+\beta) - 2\beta x^{\alpha}\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)} \right\}^{\delta} dx\right]$$

$$H_R(\delta) = \frac{1}{1-\delta} \log\left[\left\{ \frac{(\alpha+\omega)(2\alpha+\omega)}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)} \right\}^{\delta} \int_0^{\theta} x^{\delta(\alpha+\omega-1)}\{\theta^{\alpha}(1+\beta) - 2\beta x^{\alpha}\}^{\delta} dx\right]$$

Now, on using Newton's generalized binomial theorem and after some simplification we obtained.

$$H_R(\delta) = \frac{1}{1-\delta} \log\left[\left\{ \frac{(\alpha+\omega)(2\alpha+\omega)}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)} \right\}^{\delta} \sum_{k=0}^{\infty} \binom{\delta}{k} \{\theta^{\alpha}(1+\beta)\}^{\delta-k} (-2\beta)^{k} \frac{\theta^{\delta(\alpha+\omega-1)+\alpha k+1}}{\delta(\alpha+\omega-1)+\alpha k+1}\right]$$

However, the Shanon entropy $(H_S)$ which is the limiting form of Renyi entropy (i.e. $H_S = \lim_{\delta \to 1} H_R(\delta)$ ) is not obtained in the closed form but has been computed numerically and is given in Table-1.

Table 1: Characteristics of WTPD at different values of parameters

| Parameters | | | | Mean | Variance | C.V. | Skewness | Kurtosis | $H_R(\delta)$ | | | Shannon Entropy |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | $\delta$ | | | |
| $\theta$ | $\alpha$ | $\beta$ | $\omega$ | $\mu$ | $\sigma^2$ | $c.v.$ | $\gamma_1$ | $\gamma_2$ | 0.3 | 0.6 | 0.9999 | $H_S$ |
| | | | 1 | 69.697 | 505.969 | 0.1041 | -0.7024 | -0.338 | 4.531 | 4.421 | 4.3433 | 4.3429 |
| | 1 | -0.3 | 2 | 76.957 | 338.563 | 0.0571 | -0.9589 | 0.3752 | 4.370 | 4.228 | 4.1028 | 4.1027 |
| | | | 1 | 62.963 | 591.221 | 0.1491 | -0.3983 | -0.815 | 4.551 | 4.512 | 4.4740 | 4.4740 |
| | | 0.3 | 2 | 72.353 | 412.111 | 0.0787 | -0.7209 | -0.219 | 4.439 | 4.339 | 4.2524 | 4.2524 |
| | | | 1 | 92.840 | 49.4668 | 0.0057 | -1.7809 | 4.1204 | 3.582 | 3.224 | 2.9680 | 2.9679 |
| | | -0.3 | 2 | 93.373 | 42.5092 | 0.0048 | -1.8041 | 4.2983 | 3.515 | 3.148 | 2.8907 | 2.8907 |
| | 10 | | 1 | 90.459 | 65.4479 | 0.0079 | -1.3670 | 2.2390 | 3.706 | 3.427 | 3.2394 | 3.2394 |
| 100 | | 0.3 | 2 | 91.182 | 56.9231 | 0.0068 | -1.3962 | 2.3798 | 3.643 | 3.355 | 3.1624 | 3.1623 |
| | | | 1 | 96.115 | 15.7516 | 0.0017 | -1.9858 | 5.5227 | 3.055 | 2.639 | 2.3566 | 2.3565 |
| | 20 | -0.3 | 2 | 96.279 | 14.4331 | 0.0015 | -1.9923 | 5.5890 | 3.014 | 2.596 | 2.3136 | 2.3135 |
| | | | 1 | 94.784 | 21.1319 | 0.0023 | -1.5558 | 3.2763 | 3.187 | 2.854 | 2.6418 | 2.6417 |
| | | 0.3 | 2 | 95.007 | 19.4876 | 0.0021 | -1.5648 | 3.3255 | 3.148 | 2.812 | 2.5986 | 2.5986 |
| | | | 1 | 97.334 | 7.63775 | 0.0008 | -2.0677 | 6.1439 | 2.710 | 2.272 | 1.9797 | 1.9796 |
| | 30 | -0.8 | 2 | 97.412 | 7.18319 | 0.0008 | -2.0702 | 6.1772 | 2.681 | 2.242 | 1.9499 | 1.9498 |
| | | | 1 | 96.412 | 10.2977 | 0.0011 | -1.6313 | 3.7399 | 2.846 | 2.491 | 2.2697 | 2.2696 |
| | | 0.8 | 2 | 96.518 | 9.72715 | 0.0010 | -1.6353 | 3.7629 | 2.818 | 2.462 | 2.2397 | 2.2397 |

From Table-1, it can be seen that the coefficient of variation ( $c.v.$ ) is $\leq 1$ justifying that WTPD belongs N.B.U.E. family, see [10]. From coloumn (8) of the same table it is clear that WTPD is negatively skewed distribution which is also apparent from density plots given in Fig.1a. Also the last two columns of the table justify that Shannon entropy ( $H_S$ ) is the limiting case of Renyi entropy ( $H_R(\delta)$ ) as the order ( $\delta$ ) of $H_R(\delta)$ tends to 1.

### 3.3 Order Statistics

Let $x_{(1)}, x_{(2)}, x_{(3)}, \ldots, x_{(n)}$ be an ordered sample of size $n = 2m+1, m = 0,1,2,\ldots$ from WTPD. Then the *p.d.f.* of $X_{(1)}, X_{(n)}, X_{(r)}$ and $X_{(m+1)}$ (sample median) are respectively given by (3..11), (3..12), (3..13) and (3..14).

$$f_{X_{1:n}}(x) = n[1-F_\omega(x)]^{n-1} f_\omega(x)$$

$$f_{X_{1:n}}(x) = n\left[1-\frac{x^{\alpha+\omega}\{\theta^\alpha(1+\beta)(2\alpha+\omega)-2x^\alpha\beta(\alpha+\omega)\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)}\right]^{n-1} \frac{(\alpha+\omega)(2\alpha+\omega)\, x^{\alpha+\omega-1}\{\theta^\alpha(1+\beta)-2\beta x^\alpha\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)} \quad (3..11)$$

$$f_{X_{n:n}}(x) = n\,[F_\omega(x)]^{n-1} f_\omega(x)$$

$$f_{X_{n:n}}(x) = n\left[\frac{x^{\alpha+\omega}\{\theta^\alpha(1+\beta)(2\alpha+\omega)-2x^\alpha\beta(\alpha+\omega)\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)}\right]^{n-1} \frac{(\alpha+\omega)(2\alpha+\omega)x^{\alpha+\omega-1}\{\theta^\alpha(1+\beta)-2\beta\, x^\alpha\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)} \quad (3..12)$$

It can be seen that the density of $X_{(n)}$ is same as the density of exponentiated W.T.P.D. with exponent $n$.

$$f_{X_{r:n}}(x) = \frac{n!}{(r-1)!(n-r)!}[F_\omega(x)]^{r-1}[1-F_\omega(x)]^{n-r} f_\omega(x)$$

$$f_{X_{r:n}}(x) = \frac{n!}{(r-1)!(n-r)!}\left[\frac{x^{\alpha+\omega}\{\theta^\alpha(1+\beta)(2\alpha+\omega)-2x^\alpha\beta(\alpha+\omega)\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)}\right]^{r-1} \times \quad (3..13)$$

$$\left[1-\frac{x^{\alpha+\omega}\{\theta^\alpha(1+\beta)(2\alpha+\omega)-2x^\alpha\beta(\alpha+\omega)\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)}\right]^{n-r} \frac{(\alpha+\omega)(2\alpha+\omega)x^{\alpha+\omega-1}\{\theta^\alpha(1+\beta)-2\beta x^\alpha\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)}$$

$$f_{X_{m+1:n}}(x) = \frac{(2m+1)!}{m!\,m!}[1-F_\omega(x)]^m[F_\omega(x)]^m f_\omega(x)$$

$$f_{X_{m+1:n}}(x) = \frac{(2m+1)!}{m!\,m!}\left[1-\frac{x^{\alpha+\omega}\{\theta^\alpha(1+\beta)(2\alpha+\omega)-2x^\alpha\beta(\alpha+\omega)\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)}\right]^m \times \quad (3..14)$$

$$\left[\frac{x^{\alpha+\omega}\{\theta^\alpha(1+\beta)(2\alpha+\omega)-2x^\alpha\beta(\alpha+\omega)\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)}\right]^m \frac{(\alpha+\omega)(2\alpha+\omega)x^{\alpha+\omega-1}\{\theta^\alpha(1+\beta)-2\beta x^\alpha\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)}$$

Fig.3, depicts that the increase in order $(r)$ leads to decrease in deviation. Curve with $r = n$ corresponds to exponentiated WTPD ( i.e. density of $n^{th}$ order statistics $X_{(n)}$ from WTPD).

### 3.4 Income Distribution curve

**Theorem 3.5.** The Bonferroni and Lorenz curve for a random variable $X$ following WTPFD are respectively given by:

$$B(p) = \frac{(\omega+\alpha+1)(\omega+2\alpha+1)}{p\{2\alpha-(\beta-1)(\omega+1)\}\theta^{\omega+2\alpha+1}}\left[\frac{\theta^\alpha(1+\beta)\,q^{\alpha+\omega+1}}{\alpha+\omega+1}-\frac{2\beta\,q^{2\alpha+\omega+1}}{2\alpha+\omega+1}\right] \quad (3..15)$$

$$L(p) = \frac{(\omega+\alpha+1)(\omega+2\alpha+1)}{\theta^{\omega+2\alpha+1}\{2\alpha-(\beta-1)(\omega+1)\}}\left[\frac{\theta^\alpha(1+\beta)\,q^{\alpha+\omega+1}}{\alpha+\omega+1}-\frac{2\beta\,q^{2\alpha+\omega+1}}{2\alpha+\omega+1}\right] \quad (3..16)$$

**Proof.** Bonferroni curve of a random variable $X$ having density function $f(x)$ is given by:

$$B(p) = \frac{1}{p\mu}\int_0^q x\, f(x)\,dx;\ p\in[0,1] \quad (3..17)$$

$$L(p) = pB(p) = \frac{1}{\mu}\int_0^q x\, f(x)\, dx;\; p \in [0,1] \quad (3..18)$$

where, $q = F^{-1}(p)$ and $\mu = E[x]$. Therefore, using the definition given in (3.17), we get the Bonferroni curve of WTPD as:

$$B(p) = \frac{1}{p\mu}\int_0^q x\, f_\omega(x;\alpha,\theta,\beta,\omega)dx$$

$$B(p) = \frac{1}{p\mu}\int_0^q x\frac{(\alpha+\omega)(2\alpha+\omega)x^{\alpha+\omega-1}\{\theta^\alpha(1+\beta)-2\beta x^\alpha\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)}dx$$

$$B(p) = \frac{(\omega+\alpha+1)(\omega+2\alpha+1)}{p\{2\alpha-(\beta-1)(\omega+1)\}\theta^{\omega+2\alpha+1}}\left[\frac{\theta^\alpha(1+\beta)q^{\alpha+\omega+1}}{\alpha+\omega+1}-\frac{2\beta q^{2\alpha+\omega+1}}{2\alpha+\omega+1}\right]$$

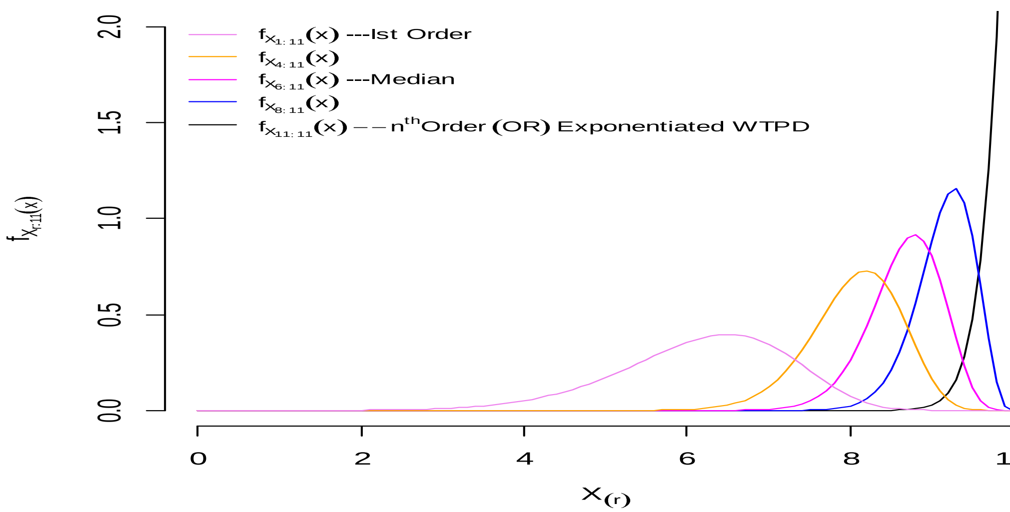



Figure 3: Density of $r^{th}$ order statistics at $\theta = 10, \alpha = 5, \beta = 0.5, \omega = 2$ and $n = 11$.

Lorenz curve is given by:

$$L(p) = p\,B(p) = \frac{(\omega+\alpha+1)(\omega+2\alpha+1)}{\{2\alpha-(\beta-1)(\omega+1)\}\theta^{\omega+2\alpha+1}}\left[\frac{\theta^\alpha(1+\beta)q^{\alpha+\omega+1}}{\alpha+\omega+1}-\frac{2\beta q^{2\alpha+\omega+1}}{2\alpha+\omega+1}\right]$$

## 4. RANDOM NUMBER GENERATION:

Random numbers from the WTPD are generated by using the inverse sampling method. In inverse sampling method, a sample of size $n$ from a particular distribution is generated by solving the equation $F(x;\Theta) = p \sim U(0,1)$ for $x$, at preassigned values of $\Theta$ and at $n$ independent values of $p$. Therefore, using the same procedures for the generation of a random number from the WTPD we follow as:

$$F_\omega(x;\alpha,\theta,\beta,\omega) = \frac{x^{\alpha+\omega}\{\theta^\alpha(1+\beta)(2\alpha+\omega)-2x^\alpha\beta(\alpha+\omega)\}}{\theta^{2\alpha+\omega}(2\alpha+\omega-\beta\omega)} = p \quad (4..1)$$

Solving (4..1) for $x$, at $n$ independent values of $p \sim U(0,1)$, we will have a required sample of size $n$ from WTPD. Solving (4..1) manually is very tedious, therefore "uniroot" function of package "stats" in R software [18] is used to solve it.

## 5. MAXIMUM LIKELIHOOD ESTIMATION:

Let $0 \le x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)} \le \theta$ be an ordered sample of size $n$ from WTPD. Therefore its log-likelihood function will be given by

$$\log\{L(\alpha,\theta,\beta,\omega \mid x)\} = \sum_{i=1}^{n} \log\{(\beta+1)\theta^{\alpha} - 2\beta x_i^{\alpha}\} + (\alpha+\omega-1)\sum_{i=1}^{n}\log(x_i) - n\log(2\alpha-\beta\omega+\omega)$$
$$- n(2\alpha+\omega)\log(\theta) + n\log(\alpha+\omega) + n\log(2\alpha+\omega) \quad (5..1)$$

Since, (5..1) is an increasing function w.r.t. $\theta$, therefore the MLE of $\theta$ for $x_{(n)} \le \theta$ is given by $\hat{\theta}_{mle} = x_{(n)}$, whereas, MLE's for the rest of three parameters $\omega, \alpha \,\&\, \beta$ is obtained by solving simultaneously the following system of score equations of (5..1) w.r.t. $\omega, \alpha \,\&\, \beta$.

$$\sum_{i=1}^{n} \frac{\{4\beta x_i^{\alpha} - (\beta+1)x_{(n)}^{\alpha}\}\{\log(x_{(n)}) - \log(x_i)\}}{(\beta+1)x_{(n)}^{\alpha} - 2\beta x_i^{\alpha}} - \frac{2n}{2\alpha-\omega+\omega\beta} + \frac{n(4\alpha+3\omega)}{(\alpha+\omega)(2\alpha+\omega)} = 0 \quad (5..2)$$

$$\sum_{i=1}^{n} \frac{2\beta(2\alpha+\omega)x_i^{\alpha} - (\beta+1)x_{(n)}^{\alpha}(\alpha+\omega)}{(\beta+1)x_{(n)}^{\alpha} - 2\beta x_i^{\alpha}} = 0 \quad (5..3)$$

$$\sum_{i=1}^{n} \frac{x_{(n)}^{\alpha}(\alpha+\omega) - (2\alpha+\omega)x_i^{\alpha}}{(\beta+1)x_{(n)}^{\alpha} - 2\beta x_i^{\alpha}} = 0 \quad (5..4)$$

The above system of equations is non-linear and obtaining the estimates in closed form is very tedious. Therefore, R software is used to obtain the numerical estimates of parameters and are given in Table-2.

## 6 APPLICATION

In application part, comparison in terms of best fit between special cases of WTPD is made after their fitting to three different types of data sets including two real life and a simulated one.

### 6.1. Real Life Data

Two real life data sets considered are light intensity and disposable income. Variable "light" and "dpi" are respectively extracted from parent data sets "star" and "savings" present in "faraway" package in R software [7], and named as light intensity and disposable income. "star" dataset consists of two columns one related to log of the surface temperature and another regarding the log of light intensity of 47 stars in the star cluster CYG OB1. The "savings" dataset consists of 50 observations on 5 variables and "dpi" is one of them giving the disposable income in dollars (\$) of 50 countries.

### 6.2. Simulated Data

Using the inverse sampling method discussed in Section (4), a sample of size $n = 100$ is generated from WTPD with $\alpha = 5, \theta = 10, \beta = -0.7$ and $\omega = 2$. Therefore, substituting $\alpha = 5, \theta = 10, \beta = -0.7$ and $\omega = 2$ in 4.1 we get:

$$F_{\omega}(x; \alpha = 5, \theta = 10, \beta = -0.7, \omega = 2) = \frac{x^7(3.6\times10^5 + 9.8x^5)}{1.06\times10^{13}} = p \quad (6..1)$$

Solving (6..1) for $x$, at 100 independent values of $p \sim U(0,1)$, we get a required sample of size 100 from

WTPD. For each value of $p$, we will have a unique solution (i.e. $x$ ) e.g. if $p = 0.25$, $p = 0.5$ and $p = 0.75$, the resulting solutions are respectively the first quartile $(Q_1)$, Median $(Q_2)$ and third quartile $(Q_3)$.

Comparison criterion AIC and Kolmogorov-Smirnov test statistic along with it's corresponding P-value is computed numerically for each of the special case after their fitting to the above mentioned data sets and are reported in Table-2.

Table 2: MLE's and different comparison criteria

| **Data** | **Distn.** | **MLE's** | | | | $-2ll$ | **AIC** | **KS-Statistic** | **P-value** |
|---|---|---|---|---|---|---|---|---|---|
| | | $\hat{\theta}_{mle}$ | $\hat{\alpha}_{mle}$ | $\hat{\beta}_{mle}$ | $\hat{\omega}_{mle}$ | | | | |
| simulated data | WTPD | 9.989 | 5.503 | -0.664 | 2.07e-6 | 214.482 | 222.482 | 0.0683 | 0.7415 |
| | ABTPD | 9.989 | 4.264 | -0.555 | $2^*$ | 215.859 | 221.859 | 0.0680 | 0.7435 |
| | LBTPD | 9.989 | 4.888 | -0.611 | $1^*$ | 215.149 | 221.149 | 0.0645 | 0.8002 |
| | TPD | 9.989 | 5.502 | -0.664 | $0^*$ | 214.482 | 220.482 | 0.0607 | 0.8551 |
| | SBPD | 9.989 | 4.239 | $0^*$ | 3.7858 | 218.865 | 222.865 | 0.0904 | 0.3875 |
| | PD | 9.989 | 8.025 | $0^*$ | $0^*$ | 218.865 | 222.865 | 0.0904 | 0.3874 |
| | SBUD | 9.989 | $1^*$ | $0^*$ | 7.0250 | 218.865 | 222.865 | 0.0904 | 0.3874 |
| | UD | 9.989 | $1^*$ | $0^*$ | $0^*$ | 265.393 | 267.434 | 0.6559 | 2.2e-16 |
| light intensity | WTPD | 6.29 | 0.014 | 0.999 | 8.2718 | 90.4719 | 98.4719 | 0.1483 | 0.2431 |
| | ABTPD | 6.29 | 4.495 | 0.931 | $2^*$ | 91.6415 | 97.6415 | 0.1437 | 0.2865 |
| | LBTPD | 6.29 | 5.290 | 0.921 | $1^*$ | 92.0313 | 98.0313 | 0.1456 | 0.2723 |
| | TPD | 6.29 | 6.107 | 0.911 | $0^*$ | 92.4625 | 98.4625 | 0.1478 | 0.2557 |
| | PD | 6.29 | 4.282 | $0^*$ | $0^*$ | 108.193 | 111.003 | 0.2603 | 0.0034 |
| | UD | 6.29 | $1^*$ | $0^*$ | $0^*$ | 109.087 | 111.087 | 0.6264 | 2.22e-16 |
| Disposable income | WTPD | 4001.89 | 0.012 | 0.997 | 1.0698 | 799.175 | 806.175 | 0.1293 | 0.3437 |
| | ABTPD | 4001.89 | 1.62e-8 | 1 | $2^*$ | 839.900 | 845.900 | 0.3712 | 4.97e-7 |
| | LBTPD | 4001.89 | 0.057 | 0.987 | $1^*$ | 799.184 | 805.184 | 0.1286 | 0.3497 |
| | TPD | 4001.89 | 0.809 | 0.859 | $0^*$ | 800.607 | 806.607 | 0.1331 | 0.3104 |
| | PD | 4001.89 | 0.574 | $0^*$ | $0^*$ | 810.743 | 814.743 | 0.1833 | 0.0609 |
| | UD | 4001.89 | $1^*$ | $0^*$ | $0^*$ | 1171.73 | 1173.73 | 0.3824 | 4.38e-7 |

| **Abbreviation** | **Full Form** |
|---|---|
| ABTPD | Area Biased Transmuted Power Distribution. |
| LBTPD | Length Biased Transmuted Power Distribution. |
| TPD | Transmuted Power Distribution. |
| SBPD | Size Biased Power Distribution. |
| PD | Power Distribution. |
| SBUD | Size Biased Uniform Distribution. |
| UD | Uniform Distribution. |

Observation with * as superscript refer to known quantities

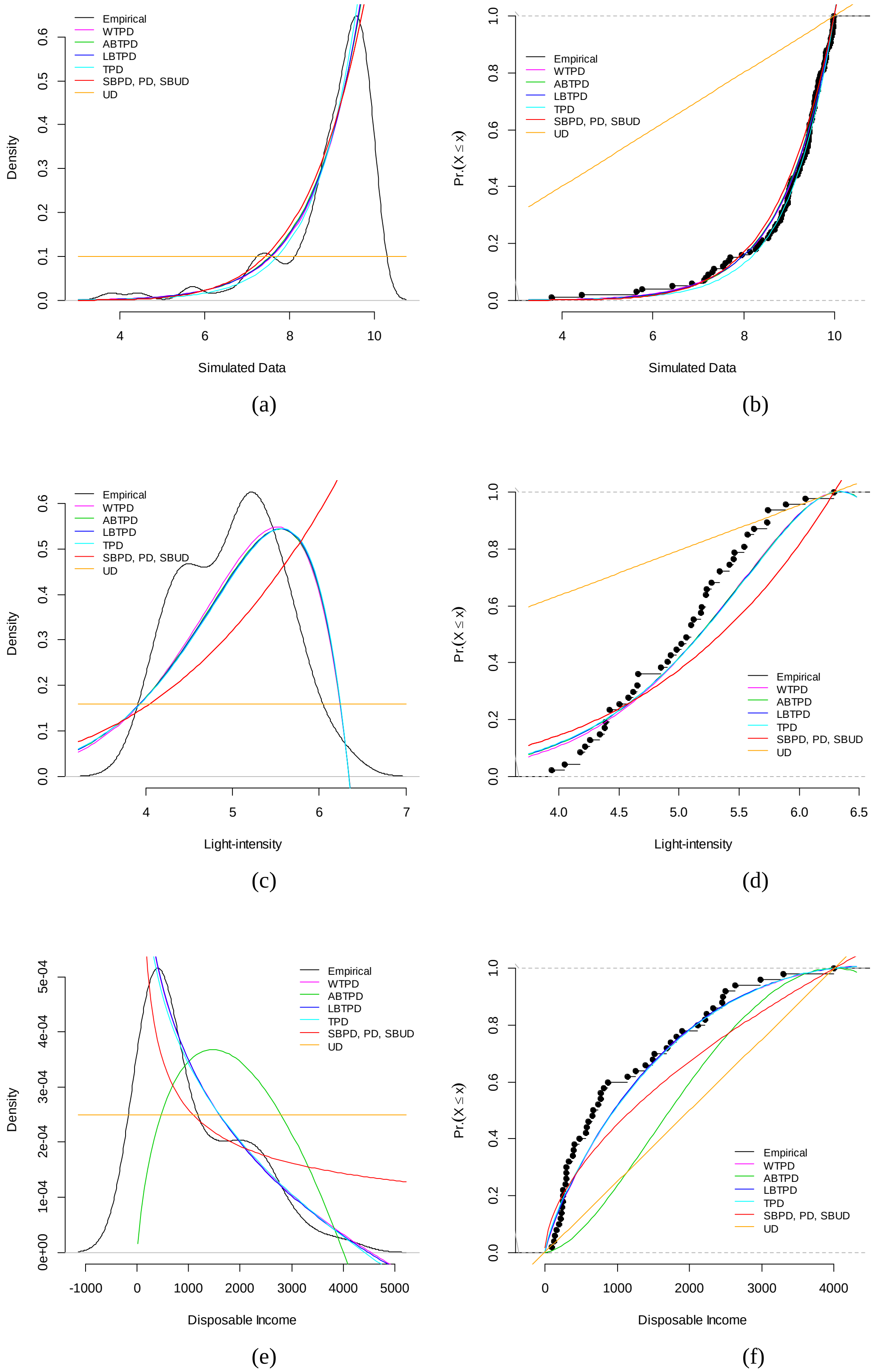


Figure 4: Empirical along with fitted density and distribution curves.

Comparing different models in terms of fitting, using the statistical tools like AIC and Kolmogorov-Smirnov test, we know that the model with minimum value of AIC, KS-Statistic or with maximum P-value is considered to be the model of best fit. Therefore, using the same interpretation, Table-2 reveals that TPD proves to be the model of best fit for simulated data set followed by LBTPD, ABTPD, PD and UD. It is also observed that SBPD, PD and SBUD provide the same result on their fitting to simulated data which justifies that PD is form-invariant and UD reduces to PD under size bias sampling. Due to the form-invariant property of PD under size bias sampling, only PD is included for comparison and the other two cases (i.e. SBPD, SBUD) are dropped for the considered real life data sets. For light intensity, it is observed that ABTPD proves to be the model of best fit followed by LBTPD, TPD, WTPD, PD and UD. Similarly, for the disposable income WTPD proves to be the best fit followed by TPD, PD, ABTPD and UD.

## 7. CONCLUSIONS

An extension of power distribution named as weighted transmuted power distribution is introduced and its different statistical properties are investigated and studied. WTPD belongs to N.B.U.E. family is justified as its coefficient of variation is $\leq 1$ and also verified graphically. The potentiality and flexibility of WTPD in expressing a random phenomenon, is supported and validated through its fitting to a simulated and two real life data sets. Inverse sampling method is used to generate the random numbers from WTPD. After the fitting of some special cases of WTPD to the considered data sets, it is concluded that TPD, ABTPD and WTPD proves to be the model of best fit respectively for the simulated data, light intensity and disposable income whereas, UD proves to be worst fit for all of the three considered data sets. The main motive behind the completion of this manuscript was to make one realize how important are the new extensions in expressing some random processes even though when we are having a number of already existing distributions. As it is observed that for the considered data sets mostly the new cases proved to be models of best fit rather than the baseline distribution i.e. power distribution.

**RECEIVED: OCTOBER, 2017**
**REVISED: JUNE, 2018**

## REFERENCES

[1] AKINSETE, A., FAMOYE, F., and LEE, C. (2008): The beta-Pareto distribution **Statistics,** 42(6):547-563.

[2] ALZAATREH, A., FAMOYE, F., and LEE, C. (2012): Gamma-Pareto Distribution and Its Applications **Journal of Modern Applied Statistical Methods**, 11(1):78-94.

[3] ARYAL, G. R. and TSOKOS, C. P. (2009): On the transmuted extreme value distribution with application **Nonlinear Analysis**, 71:1401-1407.

[4] CORDEIRO, G. M. and CASTRO, M. D. (2011): A new family of generalized distributions **Journal of Statistical Computation and Simulation**, 81(7):883-898.

[5] DAR, A. A., AHMED, A., and RESHI, J. A. (2017): Transmuted Weighted Exponential Distribution and Its Application **Journal of Statistics Applications & Probability**, 6(1):219-232.

[6] DAR, A. A., AHMED, A., and RESHI, J. A. (2018): Characterization and Estimation of Weighted Maxwell-Boltzmann Distribution **Applied Mathematics & Information Sciences**, 12(1):193-202.

[7] FARAWAY, J. (2016): **faraway: Functions and Datasets for Books by Julian Faraway R** package version 1.0.7.

[8] FISHER, R. (1934): The effect of methods of ascertainment upon the estimation of frequencies **Annals of Human Genetics**, 6(1):13-25.

[9] GUPTA, R. D. and KUNDU, D. (2001): Exponentiated Exponential Family : An Alternative to Gamma and Weibull Distributions **Biometrical Journal**, 43(1):117-130.

[10] HALL, W. J. and WELLNER., J. A. (1984): Mean residual life. **Proceedings of International Symposium on Statistics and Related Topics**, pages 169-184.

[11] HUSSIAN, M. A. (2014): Transmuted Exponentiated Gamma Distribution: A Generalization of the Exponentiated Gamma Probability Distribution **Applied Mathematical Sciences**, 8(27):1297-1310.

[12] JONES, M. C. (2009): Kumaraswamy's distribution: A beta-type distribution with some tractability advantages **Statistical Methodology**, 6(1):70-81.

[13] KVAM, P. (2008): Length bias in the measurements of carbon nanotubes **Technometrics**, 50(4):462-467.
[14] MEROVCI, F., ALIZADEH, M., YOUSOF, H. M., and HAMEDANI, G. G. (2017): The exponentiated transmuted-G family of distributions: Theory and applications **Communications in Statistics - Theory and Methods**, 46(21):10800-10822.
[15] MOBARAK, M. A., NOFAL, Z., and MAHDY, M. (2017): On Size-Biased Weighted Transmuted Weibull Distribution **International Journal of Advanced Research in Computer Science and Software Engineering**, 7(3):317-325.
[16] NADARAJAH, S., TEIMOURI, M., and SHIH, S. H. (2014): Modified Beta Distributions **Sankhya: Series B**, 76(1):19-48.
[17] NORMAN, L., KOTZ, S., and BALAKRISHNAN, N. (1994): **Continuous Univariate Distributions** Wiley, New York.
[18] R CORE TEAM (2017): **R: A Language and Environment for Statistical Computing R** Foundation for Statistical Computing, Vienna, Austria.
[19] RAO, C. R. (1965): On discrete distributions arising out of methods of ascertainment **Sankhya: Series A**, 27(2):311-324.
[20] SHAW, W. T. and BUCKLEY, I. R. C. (2009): The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map, arXiv preprint, arXiv:0901.0434.
[21] TORABI, H. and MONTAZERI, N. H. (2012): The gamma-uniform distribution and its applications **Kybernetika**, 48(1):16-30.