MULTI-OBJECTIVE LOCATION PROBLEMS WITH VARIABLE DOMINATION STRUCTURE

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ABSTRACT
We introduce an inverse variational inequality to investigate the solutions of a multi-objective optimization problem with respect to variable domination structure. In particular, mild conditions are established to study the relationships between nondominated solutions and the solutions of the inverse variational inequality. Applying these results, we study a mathematical model for location problems in the case that different preferences with respect to different facilities are at hand. Finally, we adjust a new version of PROMETHEE method to obtain solutions for a scalarized multi-objective location problem.

KEYWORDS: Multi-objective programming problem, Variable domination structure, Location problem, Weight function, PROMETHEE method.

MSC: 90C29, 90C26.

RESUMEN
En este trabajo se estudian las soluciones de un problema de optimización multi-evaluada con respecto a una estructura de dominancia variable. Bajo condiciones débiles, se establecen relaciones entre puntos no dominados y soluciones de desigualdades variacionales inversas. Estos resultados se aplican al caso de problemas de localizacón en el caso en que las preferencias son diferentes con respecto a las distintos puntos que se quieren situar. Se adapta el metodo PROMETHEE para obtener soluciones para el modelo de localización multi-objetivo escalarizado.

PALABRAS CLAVE: Estructura de dominancia variable, función de pesos, métodos PROMETHEE, problema de localización, Problema multi-objetivo.

1. INTRODUCTION

Variable domination structure, introduced by Yu [38], is nowadays instrumental to study multiobjective programming problems. Yu applied variable cones in preferences modeling [37, 38] and compared two elements in term of domination structure. Bao, Mordukhovich, Soubeyran developed mathematical models of variable domination structure in behavioral sciences [4]. They considered variable ordering cones generated by the variables pleasure and pain feeling , namely p_1 and p_2 in the decision space $\mathbb{R}^p$. Letting $d := p_1 - p_2$, they assumed that p_2 is preferred to p_1 with the domination factor d for p_1. The set of all domination factors d for p_1 together with the zero vector is denoted by $C(p_1)$ and the corresponding set valued mapping $C : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ is called variable domination structure.

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In general, the domination structure induced by the set-valued mapping $C : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ is defined by:

$$p_1 \leq_{C(p_1)} p_2,$$

if and only if

$$p_2 \in p_1 + C(p_1).$$

Analogously, one could define

$$p_1 \leq_{C(p_2)} p_2,$$

if and only if

$$p_1 \in p_2 - C(p_2).$$

To study vector objective programming problems related to variable domination structure, the concepts of minimal points and nondominated elements have been introduced in [38]. Eichfelder [17, 18] discussed these concepts in variable ordering structure and applied them in medical engineering. Soleimani and Tammer [33] introduced the concepts of approximate minimal, approximate nondominated and approximate minimizer to study vector optimization problems in the framework of the variable ordering structure.

Location decisions are now a major part of operations research and managements. It is a branch of operations research related to locating or positioning at least a new facility among several existing facilities to optimize (minimize or maximize) at least one objective function (like cost, profit, travel distance, service, waiting time and market shares). We refer the reader to [39] for a survey of recent works in this field. In [24, 34, 35] mathematical models of location problems are practically proposed as well.

The aim of the present paper is to study location problems by using a different definition of variable domination structure, which is described in the following.

In classical decision making, first a set of preferences is fixed and an appropriate weight vector is chosen, accordingly. Using this fixed weight vector, the costs of two arbitrary alternatives are compared, see for instance [8, 13, 39] and references therein. However, in many situations, every alternative (say x) has its own preferences (say $C(x)$). Therefore, it is natural that the space of decision variables to be considered as the domain space of the variable domination structure. More precisely, for the objective function $f : \mathbb{R}^n \to \mathbb{R}^p$, one may assume the variable domination structure $C : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ and compare the values of f at the points x_1 and x_2 by:

$$f(x_1) \leq_{C(x_1)} f(x_2)$$

if and only if

$$f(x_2) \in f(x_1) + C(x_1). \tag{1}$$

We put forward the claim that this variable ordering structure is very useful in location problems, strategic management, economics, organization decisions and etc. For a simple example, suppose that a factory owner intends to chose a location to install a new branch in such a way that two criteria are fulfilled: the distance to the suppliers (as minimum as possible) and the price (as low as possible). Suppose that the owner has only two possibilities x_1 and x_2 and there are only two suppliers a_1 and a_2. The owner prefers to choose the location x_1 because it is cheaper than x_2, while the location x_2 is also privileged because it is closed to suppliers a_1 and a_2. In this situation, instead of assuming a fixed vector weight, one may assume a certain weight for every criteria with respect to an alternative and apply (1) to compare the cost values mutually. It is worthy mentioning here that this kind of domination structure has been applied to study set-valued optimization problems in [16].

In this paper, we study multiobjective location problem with variable domination structure by applying an inverse variational inequality problem. We establish mild conditions to obtain the relationships between their solutions.

This paper is organized as follows: A general mathematical model with respect to the variable domination structure and a related inverse variational inequality are introduced in Section 2. In Section 3, the relationships between the solutions of multiobjective location problem and the solutions of the related inverse variational inequality are presented. Using a finite set of alternatives, we applied the obtained results to study multi-criteria decision making procedures in Section 4. A PROMETHEE method is also adjusted for the case that each criterion has (possibly) different preferences with respect to any alternative.

2. PRELIMINARIES

Consider the function $f : \mathbb{R}^n \to \mathbb{R}^p$ defined by

$$f(x) := \begin{pmatrix} f_1(x) \\ \dots \\ f_p(x) \end{pmatrix}, \tag{2}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$, for $i = 1, ..., p$. Let $C : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set valued mapping, which is called *domination map* in the sequel. The corresponding vector optimization problem with respect to the variable ordering structure generated by C is given by

$$\text{Min}(f(X), C(\cdot)), \tag{$P_{C(\cdot)}$}$$

in which $f(X) := \bigcup_{x \in X} f(x)$. In the sequel of this paper, a solution for the problem $(P_{C(\cdot)})$ is understood under the following definition (compare to [16], Definition 2.1).

Definition 2.1. *Let $x^0 \in X$. Corresponding to problem $P_{C(\cdot)}$, we say that:*

(i) *$f(x^0)$ is a minimal element of $f(X)$ with respect to the domination map C if*

$$[f(x^0) - (C(x^0) \setminus \{0\})] \cap f(X) = \emptyset.$$

The set of all minimal elements is denoted by $\mathrm{M}(f(X), C(\cdot))$.

(ii) *$f(x^0)$ is a nondominated element of $f(X)$ with respect to the domination map C if*

$$[f(x^0) - (C(x) \setminus \{0\})] \cap \{f(x)\} = \emptyset, \qquad \forall x \in X.$$

The set of all nondominated elements is denoted by $\mathrm{N}(f(X), C(\cdot))$.

(iii) $f(x^0)$ *is a weakly nondominated element of* $f(X)$ *with respect to the domination map* C *if*

$$[f(x^0) - \operatorname{int} C(x)] \cap \{f(x)\} = \emptyset, \qquad \forall x \in X.$$

In this case, one need to assume $\operatorname{int} C(x) \neq \emptyset$*, for all* $x \in X$*. The set of all weakly nondominated solutions is denoted by* $\mathrm{WN}(f(X), C(\cdot))$.

The following inverse vector variational inequality problem is introduced in order to study the nondominated and minimal elements of $(P_{C(\cdot)})$. Let $\beta : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be an arbitrary function. Determine $x^0 \in X$ such that

$$\forall x \in X : \qquad \beta(x)^T f(x) \geq \beta(x)^T f(x^0). \tag{P_β}$$

In [11], a slightly different inverse vector variational inequality problem was presented in order to study minimal elements of $(P_{C(\cdot)})$, for the objective function $f : \mathbb{R}^n \to \mathbb{R}^n$ and the domination map $C : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. In the next section, we describe the relationships between the solutions of (P_β) and nondominated elements of $(P_{C(\cdot)})$.
Assume that $X \subseteq \mathbb{R}^n$ and X is divided into a finitely many disjoint parts. More precisely, let $m \in \mathbb{N}$ be fixed and $X := \bigcup\limits_{k=1}^{m} X_k$ where $X_k \subseteq \mathbb{R}^n$ for $k = 1, ..., m$ and $X_s \cap X_j = \emptyset$ whenever $s \neq j$. Let $\alpha^k := (\alpha_1^k, ..., \alpha_p^k)^T \in \mathbb{R}^p$ for $k = 1, \cdots, m$. Define the domination map $C : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ by

$$C(x) := \begin{cases} \{(y_1, y_2, ..., y_p)^T \in \mathbb{R}^p \mid \alpha_1^1 y_1 + \alpha_2^1 y_2 + ...\alpha_p^1 y_p \geq 0\}; & \text{if } x \in X_1, \\ \cdots & \\ \{(y_1, y_2, ..., y_p)^T \in \mathbb{R}^p \mid \alpha_1^m y_1 + \alpha_2^m y_2 + ...\alpha_p^m y_p \geq 0\}; & \text{if } x \in X_m. \end{cases} \tag{3}$$

As an application in multi-objective location problems, X is considered as a region divided by m disjoint districts $X_1, \ldots, X_m$ in Section 4. The coefficients $\alpha^1, \ldots, \alpha^m$ are chosen by decision maker with respect to the characterization and preferences of each X_k, for $k = 1, \cdots, m$. In other words, α_i^j is the weight multiplier for the criterion f_i with respect to the alternative X_j and so $\alpha^j = (\alpha_1^j, ..., \alpha_p^j)^T$ is the weight vector corresponding to the district X_j.

3. NONDOMINATED SOLUTIONS OF VECTOR OPTIMIZATION PROBLEMS AND INVERSE VARIATIONAL INEQUALITIES

In this section we establish conditions to study the relationships between nondominated and minimal elements of $(P_{C(\cdot)})$ and the solutions of (P_β). For the sake of simplicity, we formalize the foregoing assumptions by Assumption 3.1 as follows:

Assumption 3.1. *Assume that* $f : \mathbb{R}^n \to \mathbb{R}^p$ *is a vector function with the components* $f_1, \ldots, f_p$*. Assume that* m *is a positive integer number,* $X \subseteq \mathbb{R}^n$ *and* $X := \bigcup\limits_{k=1}^{m} X_k$ *where* $X_k \subseteq \mathbb{R}^n$ *for* $k = 1, ..., m$ *and* $X_s \cap X_j = \emptyset$ *whenever* $s \neq j$*. Let* C *be the domination map defined by* (3)*.*

To reach our goals, we also need the following well-known separation theorem.

Theorem 3.1. (Hyperplane separation theorem.) *Let V, U be convex subsets of $\mathbb{R}^p$ with $\operatorname{int} V \neq \emptyset$. If $(\operatorname{int} V) \cap U = \emptyset$ then there exist $x' \in \mathbb{R}^p \setminus \{0\}$ and $e \in \mathbb{R}$ such that for all $v \in V$ and $u \in U$:*

$$\langle x', v \rangle \leq e \leq \langle x', u \rangle .$$

Theorem 3.2. *Let Assumption 3.1 hold. Assume that $f(x^0) \in \mathrm{M}(f(X), C(\cdot))$ for some $x^0 \in X$. If $f(X) + C(x^0)$ is convex, then there exists a constant function $\beta : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ such that x^0 solves (P_β).*

Proof. There exists $k^0 \in \{1, ..., m\}$ such that $x^0 \in X_{k^0}$. Therefore,

$$C(x^0) = \{(y_1, y_2, ..., y_p)^T \in \mathbb{R}^p | \alpha_1^{k^0} y_1 + \alpha_2^{k^0} y_2 + ... \alpha_p^{k^0} y_p \geq 0\}.$$

Using the definition of $M(f(X), C(\cdot))$, it is easy to see that

$$(f(X) + C(x^0)) \cap (f(x^0) - C(x^0) \setminus \{0\}) = \emptyset.$$

Since $C(x^0)$ is convex and has nonempty interior, $f(x^0) - C(x^0)$ is a convex subset of $\mathbb{R}^p$ and $\operatorname{int}(f(x^0) - C(x^0)) \neq \emptyset$. Applying Theorem 3.1 for $U := f(X) + C(x^0)$ and $V := f(x^0) - C(x^0)$, there exist $q := (q^1, ..., q^p) \in \mathbb{R}^p, q \neq 0$ and $\alpha \in \mathbb{R}$, such that

$$q^T v \leq \alpha \leq q^T u, \tag{4}$$

for all $v \in f(x^0) - C(x^0)$ and $u \in f(X) + C(x^0)$. Let $\beta_j^{k^0} := q^j$, for $j = 1, \cdots, p$ and define the constant function β by

$$\beta\,(x) := (\beta_1^{k^0}, \beta_2^{k^0}, ..., \beta_p^{k^0}), \qquad \forall x \in X.$$

Using (4), it follows that

$$\forall c^1, c^2 \in C(x^0), \forall x \in X : \beta^0(x)^T (f(x^0) - c^1) \leq \beta^0(x)^T (f(x) + c^2).$$

Letting $c^1 = c^2 = 0$, the desired result follows. ☐

Remark 3.1. *Let Assumption 3.1 hold. Since the function β plays the role of weight function at any point, it is required (in practice) that the values of β are nonnegative. To meet this purpose, one may assume in Theorem 3.2 that $\mathbb{R}^p_+ \subseteq C(x^0)$. Indeed, applying (4), we have for all $\lambda > 0$ and $j = 1, \cdots, p$*

$$\alpha \leq q^T (f(x^0) + \lambda e_j),$$

where $e_j \in \mathbb{R}^p_+$ is the standard unit vector whose components are 0 except the j^{th} component which is one. This implies that $0 \leq q^T e_j = q^j$, for $j = 1, \ldots, p$. In particular, for a fixed j_0, $\mathbb{R}_+ e_{j_0} \subseteq C(x^0)$ implies that $q^{j_0} \geq 0$, where $\mathbb{R}_+ e_{j_0} := \{\lambda e_{j_0} | \lambda \geq 0\}$.

Next, we study the relationship between weakly nondominated elements of $f(X)$ w.r.t. C and $(P_{C(\cdot)})$. For this end, consider the step function $\beta : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ defined by

$$\beta\,(x) := \begin{cases} \beta^1; \quad if \quad x \in X_1, \\ \cdots \\ \beta^m; \quad if \quad x \in X_m, \end{cases} \tag{5}$$

where $\beta^k \in \mathbb{R}^p$ for $k = 1, \cdots, m$.

Theorem 3.3. *Let Assumption 3.1 hold. Assume that $f(x^0) \in \mathrm{WN}(f(X), C(\cdot))$ for some $x^0 \in X$. If $f(X_i) + C(x^i)$ is convex for $i = \overline{1,m}$ and for some $x^i \in X_i$, then there exists a step function $\beta : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ defined by* (5) *such that x^0 solves* (P_β).

Proof. Let $i \in \{\overline{1,m}\}$. Since the domination map $C(.)$ is a constant convex cone on X_i, it follows that

$$(f(X_i) + C(x^i)) \cap (f(x^0) - \operatorname{int} C(x^i)) = \emptyset.$$

By a similar argument to that of Theorem 3.2, there exists a vector $\beta^i \in \mathbb{R}^p$ such that

$$\beta^{i^T}(f(x^0) - c^1) \leq \beta^{i^T}(f(x) + c^2),$$

for all $c^1 \in \operatorname{int} C(x^i), c^2 \in C(x^i)$ and $x \in X_i$. Setting $c^2 = 0$ and $\|c^1\| \to 0$, we have for all $x \in X_i$ that

$$\beta^{i^T}(f(x^0)) \leq \beta^{i^T}(f(x)).$$

Letting $\beta(x) := \beta^i$ for all $x \in X_i$, the desired result follows. □

Remark 3.2. *A similar discussion of Remark 3.1 is valid for the function β in Theorem 3.3. Indeed, if $\mathbb{R}_+ e_{j_0} \subseteq C(x^i)$ for some $j_0 \in \overline{1,p}$ and $i \in \overline{1,m}$, then the j_0^{th} component of the vector β^i of the formula* (5) *is nonnegative.*

Next, for an arbitrary domination map, we are going to establish conditions for which a solution of (P_β) implies a solution of $\mathrm{WN}(f(X), C(\cdot))$. For this end we present the following concepts.

Definition 3.2. *Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a vector function and $C : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be an arbitrary domination map. A function $\beta : \mathbb{R}^n \to \mathbb{R}^p$ is called*

(i) $C(\cdot)$-monotone on $f(X)$, if

$$\forall x^1, x^2 \in X : \qquad f(x^2) \in f(x^1) + C(x^1) \;\Rightarrow\; \beta(x^1)^T f(x^1) \leq \beta(x^1)^T f(x^2).$$

(ii) Strongly $C(\cdot)$-monotone on $f(X)$, if

$$\forall x^1, x^2 \in X : \qquad f(x^2) \in f(x^1) + (C(x^1) \setminus \{0\}) \;\Rightarrow\; \beta(x^1)^T f(x^1) < \beta(x^1)^T f(x^2)$$

(iii) Semi-strongly $C(\cdot)$-monotone on $f(X)$, if it is C-monotone on $f(X)$ and

$$\forall x^1, x^2 \in X : \qquad f(x^2) \in f(x^1) + \operatorname{int} C(x^1) \;\Rightarrow\; \beta(x^1)^T f(x^1) < \beta(x^1)^T f(x^2)$$

These concepts are illustrated through the next simple examples.

Example 3.1. *Let the domination map be a constant set valued mapping whose value be the convex cone C and $\beta : \mathbb{R}^n \to C^+$ be a constant function, where $C^+ := \{y \in \mathbb{R}^p | y^T c \geq 0,\ \forall c \in C\}$ is the positive polar cone of C. Therefore, β is $C(\cdot)$-monotone on $f(\mathbb{R}^n)$, for an arbitrary vector function $f : \mathbb{R}^n \to \mathbb{R}^p$.*

Example 3.2. *Let $\beta : X \subseteq \mathbb{R}^n \to \mathbb{R}^p$ and Y be a subspace of $\mathbb{R}^p$. Let $f : X \to Y$ and the domination map $C : X \rightrightarrows \mathbb{R}^p$ be defined by*

$$C(x) := \{y \in Y \mid y^T \beta(x) \geq 0\}$$

for all $x \in X$. Therefore, β is semi-strongly $C(\cdot)$-monotone on $f(X)$. Moreover, let $\alpha \in \mathbb{R}^p_{++} := \{(\xi_1, \ldots, \xi_p) \in \mathbb{R}^p \mid \xi_i > 0,\ i = 1, \ldots, p\}$ and

$$D(x) := \{y \in Y \mid y^T \alpha > 0\},$$

for every $x \in X$. Therefore, β is a strongly $D(\cdot)$-monotone map on $f(X)$ as well.

In the following we present the main result of this section.

Theorem 3.4. *Let $C : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be an arbitrary domination map, $x^0 \in X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}^p$ be a function. Assume that $\beta : \mathbb{R}^n \to \mathbb{R}^p$ is a semi-strongly $C(\cdot)$-monotone function on $f(X)$ and x^0 solves (P_β). Then*

$$f(x^0) \in \mathrm{WN}(f(X), C(\cdot)).$$

Moreover, if β is strongly $C(\cdot)$-monotone on $f(X)$, then

$$f(x^0) \in \mathrm{N}(f(X), C(\cdot)).$$

Proof. By hypothesis

$$\forall x \in X : \quad \beta(x)^T(f(x^0)) \le \beta(x)^T(f(x)). \tag{6}$$

Assume that $f(x^0) \notin \mathrm{WN}(f(X), C(\cdot))$. Then, there exists $\bar{x} \in X$ with $f(\bar{x}) \in f(x^0) - \operatorname{int} C(\bar{x})$. This means that

$$f(x^0) \in f(\bar{x}) + \operatorname{int} C(\bar{x}).$$

Taking into account the semi-strong $C(\cdot) - monotonicity$ of β on $f(X)$, one has

$$\beta(\bar{x})^T(f(\bar{x})) < \beta(\bar{x})^T(f(x^0)),$$

which contradicts with (6). Hence, $f(x_0)$ is a weakly nondominated element. The rest of the proof is similar and therefore we omit it. □

4. LOCATION PROBLEMS WITH RESPECT TO VARIABLE DOMINATION STRUCTURE

4.1. Preliminaries to the location problems

A short review of different categorizes, investigations and solution methods of location problems is presented in this subsection. Afterwards, the mathematical models for these problems are shown with respect to the variable domination structure.

Although there are various classifications for these problems, the present paper divides them into multi-objective and multi-attribute location problems based on the background of their decision making approaches. In addition, since the bi-objective location problems have been taken into consideration in the literature, we studied them individually from the other k-objective location problems ($k \ge 3$).

Multicriteria decision making (MCDM) methods can be applied for all types of facility location models involving the single facility location, multiple facility location, location allocation, quadratic assignment problems, covering problems, median problems, center problems, hierarchical facility location problem, hub location problems, competitive facility location, warehouse location problems, dynamic facility location problems, location-routing, location-inventory, location-reliability and especially, location in the supply chain [39].

4.1.1. The Bi-objective location problems

Here we prepare the most recent studies on bi-objective location problem. Considering the minisum and the minimax criteria, Ohsawa [30] has studied a single facility, quadratic euclidean distance bi-criteria model in the continuous space. Nickel [29] has developed the classical bi-facility Weber problems to bi-objective ones with regional restrictions. One of the most important classes of location problems is facility location on a network. Bhaskaran and Turnquist [5] have investigated the relation between coverage objectives and transportation cost in a multi-facility locations network. For more different studies, see [20, 21, 26].

4.1.2. The k-objective location problems

This class refers to the location problems, which is categorized into four classes based on their types of objectives: demand coverage, dealing cost, profit maximization and environmental issues. They can be considered with respect to their classical operational research families such as Weber, competitive, location-allocation, location routing, network, etc [2, 12, 14, 20].

4.1.3. The multi-attribute location problems

In the multi-attribute decision making there is usually a limited number of predetermined alternatives which satisfies each objective in a specified level. Therefore, according to the problem, the decision maker selects the best solution among all alternatives. Some of the well-known multi-attribute decision making methods are ANP, AHP, ELECTRE, TOPSIS, which are applied for providing solution for location problems [9, 36]. The geographical information systems (GIS) can be sometimes used for location problems. Higgs [25] has reviewed some strives in waste management siting. In his approach multi-criteria analysis and determination have been joined with GIS to take into account the effect of the public decision making. Tuzkaya et al [36] have utilized the analytic network process (ANP) technique. This technique includes qualitative and quantitative parameters, sensitive and insensitive attributes and applies four main parameters such as benefits, risk, opportunities and cost.

4.1.4. The solution methods

Many solution methods, including both exact and approximate, have been applied to access the Pareto points of a multi-criteria location problem. Puerto and Fernandez [20] use polyhedral norms to obtain approximate solutions of a multi-criteria Weber location problems. considering risk and cost, Nema and Gupta [28] have suggested a normalized composite utility function to optimize regional hazardous waste management systems. Melachrinoudis [27] has applied a decomposition strategy to shift non-convex bi-criterion problems into bi-criterion linear sub-problems. Alumur and Kara [1] have solved an integer multi-objective programming model, in which the risk is as important as cost. For more studies on the solution methods, see [20, 26].

4.2. Location problems related to the variable domination structure

Assume that $X \subseteq \mathbb{R}^n$ is divided into a finitely many disjoint parts. Note that we assume that the dimension of the decision space is n (possibly $n \geq 2$), due to the fact that, in addition two the geographical coordinates, one may assume other parameters. For example, if the preferences of objectives with respect to locations are also depend on time. Let $m \in \mathbb{N}$ be fixed and $X := \bigcup_{k=1}^{m} X_k$ where $X_k \subseteq \mathbb{R}^n$ for $k = 1, ..., m$ and $X_s \cap X_j = \emptyset$ whenever $s \neq j$. Assume that $a^i \in \mathbb{R}^n$ for $i = \overline{1,p}$ are fixed locations. Let $\| \cdot \|$ be an arbitrary norm (usually Manhatan norm). Let

$d(x,a^i) := \left\|x - a^i\right\|$ for $i = \overline{1,p}$ and define $f : \mathbb{R}^n \to \mathbb{R}^p$ by

$$f(x) := \begin{pmatrix} d(x,a^1) \\ \dots \\ d(x,a^p) \end{pmatrix}. \tag{7}$$

In our setting, each X_j has its own preferences with respect to the location a^i , for $i = \overline{1,p}$ and $j = \overline{1,m}$. Therefore, we consider a domination map C defined by (3). Our aim is to find a location $x^0 \in \mathbb{R}^n$ in such a way that $f(x^0)$ is a (weakly) nondominated minimal element of $f(X)$ with respect to $C(\cdot)$. For this end, we consider the inverse vector variational inequality problem defined by (P_β) in Section 2.

Let $\alpha^1, \dots, \alpha^m$ be chosen by decision maker with respect to the attributes and preferences of each X_k, for $k = 1, \cdots, m$. Define both the domination map $C : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ and the function $\beta : \mathbb{R}^n \to \mathbb{R}^p$ by

$$C(x) := \begin{cases} \{(y_1, y_2, ..., y_p)^T \in \mathbb{R}^p \mid \alpha_1^1 y_1 + \alpha_2^1 y_2 + ...\alpha_p^1 y_p \geq 0\}; & \text{if } x \in X_1, \\ \cdots \\ \{(y_1, y_2, ..., y_p)^T \in \mathbb{R}^p \mid \alpha_1^m y_1 + \alpha_2^m y_2 + ...\alpha_p^m y_p \geq 0\}; & \text{if } x \in X_m, \end{cases}$$

where $\alpha^j = (\alpha_1^j, \dots, \alpha_p^j)^T$ for $j = \overline{1,m}$; and

$$\beta(x) := \begin{cases} \alpha^1; \quad if \quad x \in X_1, \\ \cdots \\ \alpha^m; \quad if \quad x \in X_m, \end{cases}$$

respectively. The following observation provides sufficient conditions to obtain nondominated and weakly nondominated solutions of the location problem in the sense of Definition 2.1.

Theorem 4.5. *Assume that f, X and $C(\cdot)$ are as above and $x^0 \in X$. Assume that for $j = \overline{1,m}$, x^0 satisfies the following inequality*

$$\sum_{i=1}^{p} \alpha_i^j d\left(x^0, a^i\right) \leq \sum_{i=1}^{p} \alpha_i^j d\left(x, a^i\right), \tag{8}$$

for all $x \in X_j$. Then $f(x^0) \in \mathrm{WN}(f(X), C(\cdot))$. Moreover, let the inequality in (8) be strict and $\alpha_i^j > 0$ for $j = \overline{1,m}$ and $i = \overline{1,p}$. Then $f(x^0) \in \mathrm{N}(f(X), C(\cdot))$.

Proof. Clearly, $C(x) = \{y \in \mathbb{R}^n \mid y^T\beta(x) \geq 0\}$ for all $x \in X$. Hence, the result follows from Theorem 3.4 and Example 3.2. □

We illustrate our results through the following example.

Example 4.3. *A factory owner decides to establish a new branch in one of four candidate locations x^1, x^2, x^3 and x^4, which has the minimum distance from ten suppliers $a^1, \dots a^{10}$. The owner has different preferences for every location with respect to a supplier. The following table depicts these preferences:*

	a^1	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}
x^1	0.02	0.08	0.3	0.03	0.17	0.26	0.02	0.07	0.02	0.03
x^2	0.2	0.05	0.03	0.14	0.17	0.02	0.12	0.12	0.08	0.07
x^3	0.08	0.15	0.04	0.15	0.25	0.05	0.1	0.03	0.05	0.1
x^4	0.09	0.08	0.13	0.02	0.08	0.13	0.01	0.12	0.09	0.25

Consider each row of this table as a vector of $\mathbb{R}^{10}$ *and denote the* i^{th} *component of the* j^{th} *row by* α_i^j*, for* $j = \overline{1,4}$ *and* $i = \overline{1,10}$.

The following data are at hand from the Google map. Notice that in calculations, every component of the following geographical coordinates must be multiplied by the factor 10^6.
The geographical coordinates for suppliers:
$a^1 = (51.523454, 11.889969), a^2 = (51.513199, 11.920868), a^3 = (51.495462, 11.892029),$
$a^4 = (51.481138, 11.874863), a^5 = (51.471516, 11.904388), a^6 = (51.425299, 11.985413),$
$a^7 = (51.449483, 12.022491), a^8 = (51.447558, 12.024895), a^9 = (51.465314, 12.000175),$
$a^{10} = (51.499950, 12.030045).$
The geographical coordinates for locations:
$x^1 = (51.487552, 11.916748), x^2 = (51.481138, 11.969963), x^3 = (51.450767, 11.983696),$
$x^4 = (51.451409, 11.942154).$
The distances between alternatives and suppliers are calculated by Manhattan norm:

$$f(x) = \begin{pmatrix} f_1(x) \\ \dots \\ f_{10}(x) \end{pmatrix} = \begin{pmatrix} \|x - a^1\|_1 \\ \dots \\ \|x - a^{10}\|_1 \end{pmatrix},$$

where $\|z\|_1 := \sum_{i=1}^{p} |z_i|$ *for* $z \in \mathbb{R}^p$. *Applying Theorem 4.5, the following data shows that* x^1 *is a nondominated solution with respect to the variable domination structure generated by the preferences.*

$$\sum_{i=1}^{10} \alpha_i^2 d\left(x^2, a^i\right) = 8.4767e + 004; \qquad \sum_{i=1}^{10} \alpha_i^3 d\left(x^3, a^i\right) = 1.0740e + 005;$$

$$\sum_{i=1}^{10} \alpha_i^4 d\left(x^4, a^i\right) = 8.7092e + 004; \qquad \sum_{i=1}^{10} \alpha_i^2 d\left(x^1, a^i\right) = 7.1605e + 004;$$

$$\sum_{i=1}^{10} \alpha_i^3 d\left(x^1, a^i\right) = 5.9087e + 004; \qquad \sum_{i=1}^{10} \alpha_i^4 d\left(x^1, a^i\right) = 5.6610e + 004.$$

Indeed,

$$\sum_{i=1}^{10} \alpha_i^2 d\left(x^1, a^i\right) < \sum_{i=1}^{10} \alpha_i^2 d\left(x^2, a^i\right);$$

$$\sum_{i=1}^{10} \alpha_i^3 d\left(x^1, a^i\right) < \sum_{i=1}^{10} \alpha_i^3 d\left(x^3, a^i\right);$$

$$\sum_{i=1}^{10} \alpha_i^4 d\left(x^1, a^i\right) < \sum_{i=1}^{10} \alpha_i^4 d\left(x^4, a^i\right).$$

4.3. PROMETHEE method in scaralized multi-criteria decision making

As mentioned in the previous sections, in many problems, there are different weights of criteria with respect to the different alternatives. Consider a finite set of alternatives $x^k \in X,\ k = 1, ..., m$ and a finite set of objectives $f_i(x)$, $i = 1, ..., p$. Similar to Section 2, let $\beta^k := (\beta_1^k, \beta_2^k, \dots, \beta_p^k) \in \mathbb{R}^p$, where β_i^k represents the corresponding weight to the k^{th} alternative x^k with respect to i^{th} criterion $f_i(x)$, for $k = \overline{1,m}$ and $i = \overline{1,p}$. Let $f(x) := (f_1(x), \dots, f_p(x))$, $X := \{x^1, \dots, x^m\}$ and the weight function $\beta : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ defined by $\beta(x^k) := \beta^k$, for $k = \overline{1,m}$. In this section we are interested to study PROMETHEE method to obtain a solution of the following problem:

$$\min_{x \in X} \beta(x)^T f(x). \tag{9}$$

It is worthy mentioning that this method is very useful whenever m and p are large.
Let us first recall the PROMETHEE method (Performance Ranking Organization Methods for Enrichment Evaluation), which is used to solve MADM problems [7]. Let $d_{ab|i} := f_i(a) - f_i(b)$ and H be a usual positive non-decreasing preference function defined by:

$$H(d) := \begin{cases} 0, & d \le 0; \\ \\ 1, & d > 0. \end{cases}$$

Notice that there are various types of preference functions such as Quasi criterion, V-sharp criterion, Level criterion, Linear criterion and Gaussian criterion; see [31] and references therein. Applying the notion of preference function, one may translate the difference into a unicriterion preference degree as follows:

$$g_i(a,b) := H\left(d_{ab|i}\right),$$

for $i = \overline{1,p}$ and for each pair of alternatives $a, b \in X$. Let $\pi(a,b)$ denote the preference index for all the criteria:

$$\pi(a,b) := \sum_{i=1}^{p} \lambda_i g_i(a,b), \tag{10}$$

where λ_i represents the weight of criterion f_i. The sum of indices $\pi(a,b)$, $\phi^+(a) := \frac{1}{1-p} \sum_{b \in X} \pi(a,b)$, is called *leaving flow* which is indicating the preference of alternative a over all the others. It shows how good is the alternative a. The sum of indices $\pi(b,a)$, $\phi^-(a)$ is defined by

$$\phi^-(a) := \frac{1}{1-p} \sum_{b \in X} \pi(b,a).$$

This function indicates the preference of all other alternatives compared to a. It is called *entering flow* and shows how inferior is the alternative a.
According to PROMETHEE I, the alternative a is preferred to the alternative b if $\phi^+(a) \ge \phi^+(b)$ and $\phi^-(a) \le \phi^-(b)$. However, by PROMETHEE II, the *net outranking flow* ϕ ($\phi(a) := \phi^+(a) - \phi^-(a)$) is applied. In this case, the alternative a is preferred to the alternative b if $\phi(a) > \phi(b)$.

We now return to the main topic of this section. If the weights of criteria are dependent on alternatives, the previously described PROMETHEE could not be applied. With a slightly change of the definition of $d_{ab|i}$, we obtain a new method to solve the problem defined in (9). Using the definition of H, (10) and the fact that weights are positive (for a zero weight one could delete the related criteria), one may multiple every $d_{ab|i}$ by the corresponding weight for the criteria f_i. Let

$$\bar{d}_{x^j x^k|i} := \beta_i^j (f_i(x^j) - f_i(x^k)),$$

for $j,k = \overline{1,m}$ and $i = \overline{1,p}$ and

$$\bar{g}_i(a,b) := H\left(\bar{d}_{ab|i}\right).$$

Define now $\bar{\pi}(x^j, x^k)$ by

$$\bar{\pi}(x^j, x^k) = \sum_{i=1}^{p} \bar{g}_i(x^j, x^k).$$

Using $\bar{\pi}$, one may define ϕ^+, ϕ^- and ϕ and use both PROMETHEE I and PROMETHEE II, similarly. Note that unlike the classical PROMETHEE method, $\bar{\pi}(x^j, x^k)$ is not necessary in $[0, 1]$, however it does not affect to the proposed PROMETHEE method.

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