ON VECTOR GENERALIZED SEMI-INFINITE PROGRAMMING

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ABSTRACT
This paper deals with vector optimization problems where the feasible set is given by a generalized semi-infinite structure. We present necessary and sufficient optimality conditions both for (locally) (weakly) efficient points as well as for properly efficient points.

KEYWORDS: vector optimization, generalized semi-infinite programming, necessary and sufficient optimality conditions, (locally)(weakly) efficient point, properly efficient point.

MSC: 90C29, 90C34, 90C46, 65K05.

RESUMEN
Este artículo trata sobre problemas de optimización vectorial donde el conjunto de factibilidad viene dado por una estructura semi-infinita generalizada. Presentamos condiciones de optimailidad necesarias y suficientes tanto para puntos (localmente) (débilmente) eficientes como para puntos eficientes propios.

PALABRAS CLAVE: Optimización vectorial, optimización semi-infinita generalizada, condiciones necesarias y suficientes de optimalidad, punto (localmente)(débilmente) eficiente, punto propiamente eficiente

1. INTRODUCTION

This paper deals with non-linear vector (or multi-objective) optimization problems whose feasible sets have a generalized semi-infinite structure. That means, that all appearing functions are real-valued and defined on a finite dimensional space and that there are
• finitely many objective functions

$$f_i : x \in \mathbb{R}^n \longmapsto f_i(x) \in \mathbb{R},\ i \in P := \{1, \dots, p\},$$

• infinitely many inequality constraints

$$G : (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \longmapsto G(x, y) \in \mathbb{R}$$

where the index $y \in \mathbb{R}^m$ is varying in an (infinite) index set

$$Y(x) = \{y \in \mathbb{R}^m \mid v_l(x, y) \geq 0,\ l \in L\}$$

with a finite set L and

$$v_l : (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \longmapsto v_l(x, y) \in \mathbb{R}.$$

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Each $\overline{y} \in Y(x)$ represents a corresponding inequality constraint $G(x, \overline{y}) \geq 0$.

There are many applications both for vector optimization and generalized semi-infinite programming (see e.g. [2, 5, 6]); both areas, their theory and numerical approaches became a very active research area in recent decades.

Our problem under consideration is called a Vector Generalized Semi-Infinite Programming Problem (VGSIP) and is defined as

$$VGSIP: \quad \text{"min"} \ (f_1(x), \ldots, f_p(x)) \text{ s.t. } x \in M$$

with the feasible set

$$M = \{x \in \mathbb{R}^n \mid G(x, y) \geq 0, \quad y \in Y(x)\}$$

(the term *generalized* refers to the dependence of $Y(x)$ on x; in case that $Y(x) = Y$ is a fixed set we would have a semi-infinite structure).

Throughout this paper we assume the following:

• All appearing functions are continuously differentiable (and the corresponding gradient, e.g. for f_1 at $\overline{x}$ is a *row* vector and denoted by $Df_1(\overline{x})$).

• The set-valued mapping

$$Y : x \in \mathbb{R}^n \longmapsto Y(x) \subset \mathbb{R}$$

is upper semicontinuous at all $x \in \mathbb{R}^n$ in the sense of [1] and $Y(x)$ is compact for all $x \in \mathbb{R}^n$ (this property is sometimes also called *uniformly bounded*).

We will use the following notations. Given $z, w \in \mathbb{R}^r$ we write

- $z \leqq w$ $(z < w)$ if $z_i \leqq w_i$ $(z_i < w_i)$, $i = 1, \ldots, r$ and
- $z \leq w$ if $z_i \leqq w_i$, $i = 1, \ldots, r$ and $z \neq w$.

The origen (zero vector) in $\mathbb{R}^r$ is denoted by 0_r. As usual, the Euclidean norm is denoted by $\|\cdot\|$ and for $\overline{x} \in \mathbb{R}^r$ and a real number $\varepsilon > 0$ define $B(\overline{x}, \varepsilon) := \{x \in \mathbb{R}^r \mid \|x - \overline{x}\| < \varepsilon\}$.

Throughout this paper, let $\overline{x} \in M$ be our point under consideration. In the following definition we recall different types of solutions for vector optimization problems which are adapted straightforwardly to our problem class VGSIP.

Definition 1.1. (a) A point $\overline{x} \in M$ is called *efficient* for VGSIP if there does not exist any $x \in M$ with $f(x) \leq f(\overline{x})$.

(b) A point $\overline{x} \in M$ is called *weakly efficient for VGSIP* if there does not exist any $x \in M$ with $f(x) < f(\overline{x})$.

(c) A point $\overline{x} \in M$ is called *locally efficient for VGSIP* on $B(\overline{x}, \varepsilon)$ if there exists a real number $\varepsilon > 0$ and if there does not exist any $x \in B(\overline{x}, \varepsilon) \cap M$ with $f(x) \leq f(\overline{x})$.

(d) A point $\overline{x} \in M$ is called *locally weakly efficient for VGSIP* if there exists a real number $\varepsilon > 0$ and if there does not exist any $x \in B(\overline{x}, \varepsilon) \cap M$ with $f(x) < f(\overline{x})$.

(e) A point $\overline{x} \in M$ is called *properly efficient for VGSIP* if $\overline{x}$ is an efficient point for VGSIP and if there exists a real number K such that for any index $i \in P$ and $x \in M$ with $f_i(x) < f_i(\overline{x})$ there exists an index $j \in P$ such that $f_j(x) > f_j(\overline{x})$ and

- $K > 0$
- $\frac{f_i(\overline{x}) - f_i(x)}{f_j(x) - f_j(\overline{x})} < K$ (1.1)

The goal of this paper is as follows. On the one hand, there are many applications and several numerical approaches for vector optimization problems (cf. e.g. the standard book [2]) as well as for generalized semi-infinite programming (cf. e.g. [5, 12, 13, 14, 16]). On the other hand, there are only a few results on the relationship between vector optimization and generalized semi-infinite programming. This paper will fill in this gap to some extent by presenting necessary and sufficient optimality conditions for locally weakly efficient points for VGSIP as well as for properly efficient points for VGSIP.

In vector optimization the assignment of a so-called weighted-sum optimization problem plays an important role. The weighted-sum optimization problem assigned to VGSIP is defined as follows where the weights are $\lambda_i \geq 0$, $i \in P$, with $\lambda \geq 0_p$, $\lambda = (\lambda_1, \ldots, \lambda_p)$:

$$\min \sum_{i \in P} \lambda_i f_i(x) \text{ s.t. } x \in M. \tag{1.2}$$

A well-known relationship between VGSIP and the problem (1.2) is presented in the following proposition.

Proposition 1.1. [2] (i) If $\overline{x} \in M$ is a global (local) minimizer of the problem (1.2) for some $\lambda \geq 0_p$, then $\overline{x}$ is a (locally) weakly efficient point for VGSIP.
(ii) If $\lambda > 0_p$ in assertion (i), then $\overline{x}$ is a (locally) efficient point for VGSIP.

For $\overline{x} \in M$, the index set of active constraints is defined as

$$Y_0(\overline{x}) = \{ y \in Y(\overline{x}) \mid G(\overline{x}, y) = 0 \}.$$

By definition, each $\overline{y} \in Y_0(\overline{x})$ is a global minimizer of the related (parametric) *lower level problem*

$$\min \; G(\overline{x}, y) \text{ s.t. } y \in Y(\overline{x}),$$

(where $\overline{x}$ is the parameter vector). As usual, The *Linear Independence constraint qualification* LICQ is said to hold at $\overline{y} \in Y_0(\overline{x})$ if the gradients

$$D_y v_l(\overline{x}, \overline{y}), \; l \in L_0(\overline{x}, \overline{y}) := \{ l \in L \mid v_l(\overline{x}, \overline{y}) = 0 \}$$

are linearly independent ($D_y v_l$ refers to the partial derivative of v_l with respect to y). By the Fritz-John optimality condition [9] there exist for each $\overline{y} \in Y_0(\overline{x})$ multipliers $\overline{\alpha} \in \mathbb{R}$ and $\overline{\gamma} = (\overline{\gamma}_l, \; l \in L_0(\overline{x}, \overline{y}))$ satisfying

$$\left.\begin{array}{l} \overline{\alpha} \geq 0, \; \overline{\gamma}_l \geq 0, \; l \in L_0(\overline{x}, \overline{y}) \\ \overline{\alpha} + \sum\limits_{l \in L_0(\overline{x}, \overline{y})} \overline{\gamma}_l = 1 \\ D_y \mathcal{L}^{(\overline{x}, \overline{y})}(\overline{x}, \overline{y}, \overline{\alpha}, \overline{\gamma}) = 0_m^\top, \end{array}\right\} \tag{1.3}$$

where the latter refers to the Lagrangian

$$\mathcal{L}^{(\overline{x}, \overline{y})}(x, y, \alpha, \gamma) = \alpha G(x, y) - \sum_{l \in L_0(\overline{x}, \overline{y})} \gamma_l v_l(x, y).$$

It is shown in [10] that
• the set

$$F(\overline{x}, \overline{y}) = \left\{ (\overline{\alpha}, \overline{\gamma}) \in \mathbb{R} \times \mathbb{R}^{|L_0(\overline{x}, \overline{y})|} \mid (\overline{\alpha}, \overline{\gamma}) \text{ fulfills (1.3)} \right\}$$

is compact for $\overline{x} \in M$ and each $\overline{y} \in Y_0(\overline{x})$ and that
• the set

$$V(\overline{x}) = \bigcup_{\overline{y} \in Y_0(\overline{x})} \left\{ D_x \mathcal{L}^{(\overline{x},\overline{y})}(\overline{x}, \overline{y}, \overline{\alpha}, \overline{\gamma}) \mid (\overline{\alpha}, \overline{\gamma}) \in F(\overline{x}, \overline{y}) \right\} \tag{1.4}$$

is compact for $\overline{x} \in M$.

This paper is organized as follows. Section 2 contains some basic notations and preliminary results. In Section 3 the main results are presented: necessary and sufficient optimality conditions for a feasible point of VGSIP to be
– a (locally) weakly efficient point for VGSIP or
– a properly efficient point for VGSIP.
This paper is completed with some conclusions in Section 4.

2. NOTATIONS AND LEMMAS

In this section we recall some notations and present some known results which will be used later. Let $S \subset \mathbb{R}^n$ be a given set. Then:
• By *conv(S)* and *co(S)* we denote the convex hull of S and the convex cone hull of S, respectively, where the latter is defined as the set of all finite non-negative linear combinations of elements of S and where $co(\emptyset) = \{0\}$.
• The *contingent cone* $\Gamma^*(s, S) \subset \mathbb{R}^n$ of S at an element $s \in S$ is defined as follows: $\overline{d} \in \Gamma^*(s, S)$ if and only if there exist sequences $\{t^\nu\}_{\nu \in \mathbb{N}} \subset \mathbb{R}$ and $\{d^\nu\}_{\nu \in \mathbb{N}} \subset \mathbb{R}^n$ such that

$$t^\nu \downarrow 0, \; d^\nu \to \overline{d} \text{ and } s + t^\nu d^\nu \in S \text{ for all } \nu \in \mathbb{N}.$$

Lemma 2.1. [7][8, Lemma 3.1.12].
Let $S \subset \mathbb{R}^n$ be a compact set. Then:
(i) If $0_n \notin \text{conv}(S)$, then $\text{co}(S)$ is closed.
(ii) It is $0_n \notin \text{conv}(S)$ if and only if the system

$$w^\top d < 0, \; w \in S$$

has a solution $d \in \mathbb{R}^n$.

For $\overline{x} \in M$ we refer to the compact set $V(\overline{x})$ defined in (1.4) and define the following corresponding sets:

$$A'(\overline{x}, M) := \left\{ d \in \mathbb{R}^n \mid w^\top d \leq 0, \; w \in V(\overline{x}) \right\} \text{ and}$$
$$A_0(\overline{x}, M) := \left\{ d \in \mathbb{R}^n \mid w^\top d < 0, \; w \in V(\overline{x}) \right\}.$$

Obviously, it is $A_0(\overline{x}, M) \subset A'(\overline{x}, M)$.
The following constraint qualification was introduced in [10] and is a generalization of the well known Mangasarian-Fromovitz constraint qualification (see [11]) to generalized semi-infinite programming.

The *extended Mangasarian-Fromovitz constraint qualification (EMFCQ)* is said to hold at $\overline{x} \in M$ if there exists a vector $d \in \mathbb{R}^n$ such that

$$w^\top d < 0, \; w \in V(\overline{x})$$

(in other words, $d \in A_0(\overline{x}, M)$).

The following is an obvious corollary from Lemma 2.1. since $V(\overline{x})$ is a compact set.

Corollary 2.1. If EMFCQ holds at $\overline{x} \in M$, then $\text{co}(V(\overline{x}))$ is closed.

The next lemma states some well-known results under the assumption that EMFCQ holds at a point $\overline{x} \in M$.

Lemma 2.2. [5, Lemma 2.4] Assume that EMFCQ holds at a point $\overline{x} \in M$. Then:
(i) $-A'(\overline{x}, M) \subset \Gamma^*(\overline{x}, M)$.
(ii) If $\overline{d} \in \mathbb{R}^n$ solves the system

$$w^\top \overline{d} > 0, \ \ w \in V(\overline{x})$$

(that is, $-\overline{d} \in A_0(\overline{x}, M)$), then there exists a positive $\overline{t} \in \mathbb{R}$ such that

$$\overline{x} + t\overline{d} \in M \text{ for all } t \in [0, \overline{t}].$$

The final lemma in this section recalls a statement of alternatives for infinite systems.

Lemma 2.3. [4, Theorem 3.5] Let $S^1 \subset \mathbb{R}^n$ be a finite set and $S^2 \subset \mathbb{R}^n$ such that $\mathrm{co}(S^2)$ is closed. Then, exactly one of the following two assertions (i) and (ii) holds:
(i) $0_n \in \left\{ w^1 + w^2 \mid w^1 \in \mathrm{conv}(S^1), \ w^2 \in \mathrm{co}(S^2) \right\}$.
(ii) The system

$$\begin{aligned} (s^1)^\top d < 0, \ \ s^1 \in S^1 \\ (s^2)^\top d \leq 0, \ \ s^2 \in S^2 \end{aligned}$$

has a solution $d \in \mathbb{R}^n$.

3. OPTIMALITY CONDITIONS FOR LOCALLY WEAKLY EFFICIENT AND PROPERLY EFFICIENT POINTS

This section contains the main results of this paper. We will state necessary as well as sufficient optimality conditions for
• a (locally) weakly efficient point for VGSIP (Theorems 3.1. and 3.2.) and for
• a properly efficient point for VGSIP (Theorems 3.3. and 3.4.).
In the remainder of this paper the point $\overline{x} \in M$ will be our point under consideration and we will sometimes delete the argument $\overline{x}$, that is, Df_i means $Df_i(\overline{x})$, $i \in P$.

Theorem 3.1. Assume that EMFCQ holds at $\overline{x} \in M$ and that $\overline{x}$ is a locally weakly efficient point for VGSIP. Then,

$$\left.\begin{array}{l} \text{there exist finitely many} \\ \bullet \ y^j \in Y_0(\overline{x}), \ \ (\alpha^j, \gamma^j) \in F(\overline{x}, y^j), \ \mu_j \geq 0, \ j = 1, \ldots, q \\ \bullet \text{ as well as } \lambda_i \geq 0, \ i \in P \text{ such that} \\ \sum\limits_{i \in P} \lambda_i Df_i - \sum\limits_{j=1}^{q} \mu_j D_x \mathcal{L}^{(\overline{x}, y^j)}(\overline{x}, y^j, \alpha^j, \gamma^j) = 0_n^\top . \end{array}\right\} \quad (3.1)$$

Proof. Since $\overline{x}$ is a locally weakly efficient point for VGSIP, we get from [15] that

$$\{d \in \mathbb{R}^n \mid Df_i d < 0, \ i \in P\} \cap \Gamma^*(\overline{x}, M) = \emptyset.$$

By Lemma 2.2. (i) and the property that EMFCQ holds at $\overline{x}$, the system

$$\begin{array}{l} Df_i d < 0, \ i \in P \\ w^\top d \leq 0, \ w \in -V(\overline{x}) \ (:= \{-w \mid w \in V(\overline{x})\}), \end{array}$$

has no solution $d \in \mathbb{R}^n$. Furthermore, Corollary 2.1. implies that co $(V(\overline{x}))$ is closed. Then, Lemma 2.3. yields

$$0_n \in \left\{w^1 - w^2 \mid w^1 \in conv(\{Df_i,\ i \in P\}),\ w^2 \in \ \mathrm{co}(V(\overline{x}))\right\},$$

that is, there exist finitely many $y^j \in Y_0(\overline{x})$, $\left(\alpha^j, \gamma^j\right) \in F\left(\overline{x}, y^j\right)$, $\mu_j \geq 0$, $j = 1, \ldots, q$ and $\lambda_i \geq 0$, $i \in P$ such that (3.1) is fulfilled. This completes the proof. △

In order to obtain *sufficient* optimality conditions we will consider the special case where certain convexity assumptions are fulfilled. The following lemma refers to the convexity of the feasible set.

Lemma 3.1. [16] Assume that
- the function $G(x, y)$ is concave on $\mathbb{R}^{n+m}$ (that is, with respect to (x, y)) and that
- the following set-valued inclusion holds:

$$Y\left(\rho x^1 + (1-\rho)x^2\right) \subset \rho Y(x^1) + (1-\rho)Y(x^2)$$

for all $x^1, x^2 \in \mathbb{R}^n$ and all $\rho \in [0, 1]$.
Then, the feasible set M is convex.

Now, we state a sufficient optimality condition.

Theorem 3.2. Suppose
- that the functions f_i, $i \in P$ are convex,
- that the assumptions of Lemma 3.1. hold and
- that there exist finitely many $y^j \in Y_0(\overline{x})$, $\left(\alpha^j, \gamma^j\right) \in F\left(\overline{x}, y^j\right)$, $\mu_j \geq 0$, $j = 1, \ldots, q$ as well as $\lambda_i \geq 0$, $i \in P$ such that condition (3.1) is fulfilled.

Furthermore, let the functions $v_l(x, y)$, $l \in L$ be convex with respect to (x, y) and assume that LICQ holds at all $y^j \in Y_0(\overline{x})$, $j = 1, \ldots, q$.
Then, $\overline{x}$ is a weakly efficient point for VGSIP. In particular, $\overline{x}$ is an efficient point for VGSIP if $\lambda_i > 0$, $i \in P$.

Proof. By [16, Theorem 5], $\overline{x}$ is a global minimizer of problem (1.2). Then, by Proposition 1.1, $\overline{x}$ is a weakly efficient point (if $\lambda_i \geq 0$, $i \in P$) for VGSIP or an efficient point (if $\lambda_i > 0$, $i \in P$) for VGSIP. This completes the proof. △

By Definition 1.1. it turns out that at an efficient point for VGSIP a decrease of a function f_{i_1}, $i_1 \in P$ in a certain direction is only possible if another function f_{i_2}, $i_2 \in P$ is increasing in this direction. This so-called trade-off is described by the ratio between the decrease of f_{i_1} and the increase of f_{i_2}.

According to Definition 1.1. (e), particularly (1.1), a properly efficient point is characterized by a bounded trade-off. In the following we will consider a properly efficient point $\overline{x} \in M$ and state corresponding necessary and sufficient optimality conditions. However, before that we present two examples which illustrate the concept of a properly efficient point.

Example 3.1. The feasible set in this example is taken from [5, Example 3.4]. Let
- $n = 2$, $m = 2$, $p = 2$, $L = \{1, 2, 3, 4, 5\}$,
- $f_1(x) = -x_1$, $f_2(x) = x_2$, $G(x, y) = -y_2 + x_2 y_1^4$,
- $v_1(x, y) = -y_2 + x_1 y_1^2$, $v_2(x, y) = y_1 + 1$, $v_3(x, y) = -y_1 + 1$, $v_4(x, y) = y_2 + 1$, $v_5(x, y) = -y_2 + 1$.

A short calculation shows that $\overline{x}^\top = (0, 0) = 0_2^\top$ is an efficient point for this VGSIP. The following observation shows that 0_2 is also a properly efficient point for this VGSIP.
The points where one function f_{i_1} decreases and another function f_{i_2} increases can be described as:

$$\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2) \ \textit{with}\ \widetilde{x}_1 < 0,\ \widetilde{x}_2 \geq \widetilde{x}_1,\ \widetilde{x}_2 < 0 \ \textit{and}$$
$$\widehat{x} = (\widehat{x}_1, \widehat{x}_2) \ \textit{with}\ \widehat{x}_1 < 0,\ \widehat{x}_2 < \widehat{x}_1,\ \widehat{x}_2 \geq -\widehat{x}_1^2.$$

For $\widetilde{x}$ there exist $\delta_1 \geq 0$, $\delta_2 > 0$ such that $\widetilde{x} = (-\delta_1 - \delta_2, -\delta_2)$. Consequently, we have

$$\frac{f_2(\overline{x}) - f_2(\widetilde{x})}{f_1(\widetilde{x}) - f_1(\overline{x})} = \frac{\delta_2}{\delta_1 + \delta_2} \leq 1.$$

For $\widehat{x}$ we have

$$\frac{f_2(\overline{x}) - f_2(\widehat{x})}{f_1(\widehat{x}) - f_1(\overline{x})} = \frac{\widehat{x}_2}{\widehat{x}_1} < 1.$$

Therefore, $\overline{x} = 0_2$ is a properly efficient point for VGSIP and we can choose $K = 1$ in (1.1). △

Example 3.2. This example illustrates that an efficient point is not necessarily also properly efficient. The feasible set in this example is based on [14, Example 2.7]. Let

- $n = 2,\ m = 1,\ p = 2,\ L = \{1, 2\}$,
- $f_1(x) = x_1,\ f_2(x) = x_2,\ G(x, y) = y$,
- $v_1(x, y) = y - 1 + (x_1 - 1)^2 + (x_2 - 1)^2$,
- $v_2(x, y) = 2 - (x_1 - 1)^2 - (x_2 - 1)^2 - x_1 - y$.

It is easy to see that $\overline{x}^\top = (1, 0)$ is an efficient point for this VGSIP. The following argument is taken from [2, Example 2.40] and it will show that $(1, 0)^\top$ is not a properly efficient point.
Define for $\varepsilon \in [0, 1]$:

$$x_1(\varepsilon) := 1 - \varepsilon,\ \ x_2(\varepsilon) := 1 - \sqrt{1 - \varepsilon^2}.$$

Then, the point $x(\varepsilon)^\top = (x_1(\varepsilon), x_2(\varepsilon))$ is an efficient point but the term

$$\frac{f_1(\overline{x}) - f_1(x(\varepsilon))}{f_2(x(\varepsilon)) - f_2(\overline{x})} = \frac{\varepsilon}{1 - \sqrt{1 - \varepsilon^2}}$$

tends to infinity (it becomes unbounded) as $\varepsilon \downarrow 0$. So, the condition (1.1) in Definition 1.1. (e) is not fulfilled. △

The next lemma presents an important property of a properly efficient point.

Lemma 3.2. Let $\overline{x} \in M$ be a properly efficient point for VGSIP and assume that EMFCQ holds at $\overline{x} \in M$. Then, for each non-empty subset $I \subset P$ the system

$$\begin{aligned} Df_i d &< 0,\ \ i \in I \\ Df_j d &\leq 0,\ \ j \in P \\ w^\top d &\leq 0,\ \ w \in -V(\overline{x}) \end{aligned}$$

has no solution $d \in \mathbb{R}^n$.

Proof. Suppose the contrary and assume without loss of generality that there exists $\overline{d} \in \mathbb{R}^n$ with

$$Df_1 \overline{d} < 0, \tag{3.2}$$

$$\begin{aligned} Df_j \overline{d} &\leq 0,\ \ j = 2, \ldots, p \\ w^\top \overline{d} &\leq 0,\ \ w \in -V(\overline{x}). \end{aligned} \tag{3.3}$$

The latter means $\overline{d} \in -A'(\overline{x}, M)$ and, by Lemma 2.2. (i), we get $\overline{d} \in \Gamma^*(\overline{x}, M)$. Therefore, there exist sequences $\{t^\nu\}_{\nu \in \mathbb{N}} \subset \mathbb{R}$ and $\{d^\nu\}_{\nu \in \mathbb{N}} \subset \mathbb{R}^n$ such that $t^\nu \downarrow 0$, $d^\nu \to \overline{d}$ and $\overline{x} + t^\nu d^\nu \in M$ for all $\nu \in \mathbb{N}$. Perhaps after shrinking to an appropriate subsequence suppose for a moment that

$$f_1(\overline{x}+t^\nu d^\nu)-f_1(\overline{x})\geq 0 \text{ for all } \nu\in\mathbb{N}.$$

Then, by the mean value theorem,

$$f_1(\overline{x}+t^\nu d^\nu)-f_1(\overline{x})=t^\nu Df_1(\overline{x}+\triangle^\nu d^\nu)d^\nu\geq 0$$

for some $\triangle^\nu\in(0,t^\nu)$. For $\nu\to\infty$ we had $Df_1\overline{d}\geq 0$ which contradicts (3.2). Therefore, we have without loss of generality

$$f_1(\overline{x}+t^\nu d^\nu)<f_1(\overline{x}) \text{ for all } \nu\in\mathbb{N}.$$

Since $\overline{x}$ is a properly efficient point for VGSIP there exists a real number $K>0$ as in Definition 1.1. (e) and - perhaps after passing to a subsequence - an index $j_0\in\{2,\dots,p\}$ such that

$$\begin{aligned}&f_{j_0}(\overline{x}+t^\nu d^\nu)>f_{j_0}(\overline{x}) \text{ and}\\ &\frac{f_1(\overline{x})-f_1(\overline{x}+t^\nu d^\nu)}{f_{j_0}(\overline{x}+t^\nu d^\nu)-f_{j_0}(\overline{x})}\leq K.\end{aligned} \tag{3.4}$$

By applying the mean value theorem to the latter two inequalities and by letting $\nu\to\infty$ we obtain

$$Df_{j_0}\overline{d}\geq 0$$

and, by (3.3), $Df_{j_0}\overline{d}=0$ as well as the unboundedness of the left-hand-side of (3.4) (recall, by (3.2), that $Df_1\overline{d}<0$). However, this contradicts (3.4) and the proof is complete. △

The next theorem represents a first order necessary optimality condition for a properly efficient point where the main difference to Theorem 3.1. (which represents a first order necessary optimality condition for a locally weakly efficient point) is that all multipliers λ_i, $i\in P$ have to be positive.

Theorem 3.3. Assume that EMFCQ holds at $\overline{x}\in M$ and that $\overline{x}$ is a properly efficient point for VGSIP. Then,

$$\left.\begin{array}{l}\text{there exist finitely many}\\ \bullet\ y^j\in Y_0(\overline{x}),\ \ (\alpha^j,\gamma^j)\in F\left(\overline{x},y^j\right),\ \mu_j\geq 0,\ j=1,\dots,q\\ \bullet\ \text{as well as } \lambda_i>0,\ i\in P \text{ such that}\\ \sum\limits_{i\in P}\lambda_i Df_i-\sum\limits_{j=1}^{q}\mu_j D_x\mathcal{L}^{(\overline{x},y^j)}(\overline{x},y^j,\alpha^j,\gamma^j)=0_n^\top.\end{array}\right\} \tag{3.5}$$

Proof. We will show for each index $i_0\in P$ that

$$-Df_{i_0}\in co\ \left(\{Df_i,\ i\in P\}\cup\{-V(\overline{x})\}\right). \tag{3.6}$$

Then, a summation of the latter p particular combinations yields a combination as described in this theorem. Choose the index $i_0\in P$ arbitrarily and fixed, say $i_0=p$. We distinguish two cases.
Case 1. The system

$$w^\top d<0,\ w\in -V(\overline{x}),\ Df_p d<0 \tag{3.7}$$

has no solution.
Since the set $\{Df_p\}\cup\{-V(\overline{x})\}$ is compact, Lemma 2.1. (ii) implies that

$$0_n^\top\in\ conv\ \left(\{Df_p\}\cup\{-V(\overline{x})\}\right).$$

Since EMFCQ holds at $\overline{x}$, the latter yields that there exists a combination

$$Df_p^\top - w = 0_n$$

for some $w^\top \in co\,(V(\overline{x}))$ (recall that EMFCQ means $0_n^\top \notin co\,(V(\overline{x}))$) and, therefore, we obtain (3.6):

$$-Df_p \in co(-V(\overline{x})).$$

Case 2. The system (3.7) has a solution. Then, we can choose a subset $\widehat{P} \subset P$ such that $p \in \widehat{P}$ and the system

$$w^\top d < 0,\ w \in -V(\overline{x}),\ Df_i d < 0,\ i \in \widehat{P} \tag{3.8}$$

has a solution and that for any solution $\overline{d}$ of (3.8) we have

$$Df_j \overline{d} \geq 0,\ j \in P \setminus \widehat{P}. \tag{3.9}$$

Obviously, such a ("maximal") subset $\widehat{P}$ always exists. Note that, by Lemma 3.2. there exists an index $j_0 \in P \setminus \widehat{P}$ such that $Df_{j_0}\overline{d} > 0$.

Now, let $\widehat{d} \in \mathbb{R}^n$ be a solution of (3.8) (and, therefore, also of (3.9)).

Proposition 3.1. There exists an index $j_1 \in P \setminus \widehat{P}$ such that $Df_{j_1}\overline{d} > 0$ and

$$\frac{-Df_p\overline{d}}{Df_{j_1}\overline{d}} \leq K, \tag{3.10}$$

where $K > 0$ is a real number as described in Definition 1.1. (e).

Proof of Proposition.
According to Lemma 2.2. (ii) and a Taylor expansion there exists a positive number $\overline{t} \in \mathbb{R}$ such that for all $t \in [0, \overline{t}]$ we have

$$\begin{aligned} &\overline{x} + t\overline{d} \in M \\ &f_i(\overline{x} + t\overline{d}) < f_i(\overline{x}),\ i \in \widehat{P}, \\ &f_j(\overline{x} + t\overline{d}) > f_j(\overline{x}),\ j \in P \setminus \widehat{P} \text{ and } Df_j\overline{d} > 0. \end{aligned}$$

Since $\overline{x}$ is a properly efficient point and

$$f_p(\overline{x} + t\overline{d}) < f_p(\overline{x}),\ t \in (0, \overline{t}],$$

there exist (by Definition 1.1. (e)), without loss of generality, a sequence $\{t^\nu\}_{\nu \in \mathbb{N}} \subset \mathbb{R}$ with $t^\nu \downarrow 0$ and an index $j_1 \in P \setminus \widehat{P}$ such that

$$\frac{f_p(\overline{x}) - f_p(\overline{x} + t^\nu\overline{d})}{f_{j_1}(\overline{x} + t^\nu\overline{d}) - f_{j_1}(\overline{x})} \leq K.$$

The mean value theorem gives

$$\frac{-t^\nu Df_p(\overline{x} + \triangle_1^\nu\overline{d})\overline{d}}{t^\nu Df_{j_1}(\overline{x} + \triangle_2^\nu\overline{d})\overline{d}} \leq K$$

for $\triangle_1^\nu, \triangle_2^\nu \in (0, t^\nu)$ and for $\nu \to \infty$ we get

$$\frac{-Df_p(\overline{x})\overline{d}}{Df_{j_1}(\overline{x})\overline{d}} \leq K,$$

which completes the proof of Proposition.

Now, we are in the following situation. The vector $\overline{d} \in \mathbb{R}^n$ solves (3.8) and (3.9) and there is an index $j_1 \in P \setminus \widehat{P}$ fulfilling (3.10). Then, we obtain

$$\frac{-Df_p\overline{d}}{\sum\limits_{j\in P\setminus\widehat{P}} Df_j\overline{d}} \leq \frac{-Df_p\overline{d}}{Df_{j_1}\overline{d}} \leq K$$

and, therefore,

$$K \sum_{j\in P\setminus\widehat{P}} Df_j\overline{d} + Df_p\overline{d} \geq 0. \tag{3.11}$$

Since (3.11) holds for all solutions $\overline{d} \in \mathbb{R}^n$ of the system (3.8) we have that the set

$$\overline{S} = \left\{ K \sum_{j\in P\setminus\widehat{P}} Df_j + Df_p,\ Df_i,\ i \in \widehat{P} \right\} \cup \{-V(\overline{x})\}$$

is compact and $0_n \in conv\,(\overline{S})$ (by Lemma 2.1. (ii)). The latter and, by (3.8), $0_n \notin conv\left(\left\{Df_i,\ i \in \widehat{P}\right\} \cup \{-V(\overline{x})\}\right)$ imply that

$$-K \sum_{j\in P\setminus\widehat{P}} Df_j - Df_p \in\ co\ \left(\left\{Df_i,\ i \in \widehat{P}\right\} \cup \{-V(\overline{x})\}\right)$$

and, therefore, we obtain (3.6):

$$-Df_p \in co\ \left(\{Df_i,\ i \in P\} \cup \{-V(\overline{x})\}\right).$$

Since the index $i_0 = p$ was chosen arbitrarily and fixed, this completes the proof of the theorem. △

Finally, in order to obtain sufficient optimality conditions for properly efficient points we assume again certain convexity properties. The following lemma recalls a relationship between properly efficient points and a related weighted-sum optimization problem as defined in (1.2).

Lemma 3.3. [3] Assume that M is a convex set and that the functions f_i, $i \in P$ are convex as well. Then, $\overline{x}$ is a properly efficient point for VGSIP if and only if $\overline{x}$ is a global minimizer of the problem (1.2) for some $\lambda_i > 0$, $i \in P$.

The final theorem presents sufficient optimality conditions for $\overline{x} \in M$ to be a properly efficient point for VGSIP.

Theorem 3.4. Suppose
- that the functions f_i, $i \in P$ are convex,
- that the assumptions of Lemma 3.1. hold and
- that condition (3.5) is fulfilled.

Furthermore, let the functions $v_l(x, y)$, $l \in L$ be convex with respect to (x, y) and assume that LICQ holds at all $y^j \in Y_0(\overline{x})$, $j = 1, \ldots, q$ (as chosen in (3.5)).
Then, $\overline{x}$ is a properly efficient point for VGSIP.

Proof. By [16, Theorem 5], $\overline{x}$ is a global minimizer of problem (1.2) with strictly positive multipliers $\lambda_i > 0$, $i \in P$. Then, Lemma 3.3. implies that $\overline{x}$ is a properly efficient point for VGSIP. △

4. CONCLUSIONS

In this paper we considered the particular class VGSIP of vector optimization problems whose feasible sets have a generalized semi-infinite structure. That is, all functions are real-valued and defined on a finite-dimensional space where the index set of inequality constraints is infinite and depends on the decision variables. Although both problem classes, vector optimization and generalized semi-infinite programming, have been recently very active research topics within mathematical programming, a systematic description of properties of the combined problem class VGSIP is still missing. This paper fills in this gap to some extent by presenting necessary and sufficient optimality conditions for a feasible point to be a (locally)(weakly) efficient point for VGSIP or a properly efficient point for VGSIP.

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REFERENCES

[1] BERGE, C. (1963): **Topological Spaces**. Oliver and Boyd, London.

[2] EHRGOTT, M. (2005): **Multicriteria Optimization**. Springer, Berlin.

[3] GEOFFRION, A. (1968): Proper efficiency and the theory of vector maximization, **J. Math. Anal. Appl.** 22, 618–630.

[4] GOBERNA, M.A. and LÓPEZ, M.A. (1998): **Linear Semi-Infinite Optimization**. Wiley, New York.

[5] GUERRA-VÁZQUEZ, F. and RÜCKMANN, J.-J. (2005): Extensions of the Kuhn-Tucker constraint qualification to generalized semi-infinite programming, **SIAM J. Optimization** 15, 926–937.

[6] GUERRA-VÁZQUEZ, F., RÜCKMANN, J.-J., STEIN, O. and STILL, G. (2008): Generalized semi-infinite programming: a tutorial. **J. Comp. Appl. Mathematics** 217, 394–419.

[7] HETTICH, R. and KORTANEK, K.O. (1993): Semi-infinite programming: theory, methods and applications, **SIAM Review** 35, 380–429.

[8] HETTICH, R. and ZENCKE, P. (1982): **Numerische Methoden der Approximation und semi-infiniten Optimierung**. Teubner. Stuttgart.

[9] JOHN, F. (1948): Extremum problems with inequalities as subsidiary conditions. In: FRIEDRICHS, K. O., NEUGEBAUER, O. E. and STOKER, J. J. (eds.) **Studies and Essays Presented to R. Courant on His 60th Birthday**, 187–204. Interscience Publishers, New York.

[10] JONGEN, H.TH., RÜCKMANN, J.-J. and STEIN, O. (1998): Generalized semi-infinite optimization: A first order optimality condition and examples, **Mathematical Programming** 83, 145–158.

[11] MANGASARIAN, O. L. and FROMOVITZ, S. (1967): The Fritz John necessary optimality conditions in the presence of equality and inequality constraints, **J. Math. Anal. Appl.** 17, 37–47.

[12] RÜCKMANN, J.–J. and SHAPIRO, A. (1999): First order optimality conditions in generalized semi–infinite programming. **J. Optimization Theory Appl.** 101, 677–691.

[13] RÜCKMANN, J.–J. and SHAPIRO, A. (2001): Second order optimality conditions in generalized semi–infinite programming. **Set–Valued Analysis** 9, 169–186.

[14] RÜCKMANN, J.-J. and STEIN, O. (2001): On linear and linearized generalized semi-infinite optimization problems, **Annals of Operations Research** 101, 191–208.

[15] SINGH, C. (1987): Optimality conditions in multiobjective differentiable programming, **J. Optimization Theory Appl.** 53, 115–123.

[16] STILL, G. (2001): Generalized semi-infinite programming: numerical aspects, **Optimization** 49, 223–242.