AN OPTIMIZATION METHOD FOR THE RECONSTRUCCTION OF THE CRYSTALLINE GRIN

Roberto Carlos Cruz-Rodríguez*[1], Fernando Raúl Rodríguez-Flores**, Alfo José Batista-Leyva*
*Instituto Superior de Tecnologías y Ciencias Aplicadas, Universidad de La Habana, La Habana, Cuba
**Facultad de Matemática y Computación, Universidad de La Habana, La Habana Cuba

ABSTRACT
In this paper, we present a method for obtaining the parameters characterizing the distribution of refractive index (GRIN) of a crystalline from measurements of the path of a light ray before and after passing through the isolated lens. The problem to be solved is addressed as an optimization one. We describe the procedure to calculate the ray path through the crystalline, based on the variational principle of Fermat, as a fundamental part of the method. To solve the optimization problem SIMPLEX (Nelder & Mead) and quasi-Newton (BFGS) methods are used, combined with the global search algorithm BASIN-HOPPING. To evaluate the method, the GRIN biparabolic model is used. We analyze the influence of the experimental uncertainties in the measurements of the ray paths and the number of measurements on the accuracy of the reconstruction. The results show that the method allows the GRIN reconstruction with acceptable accuracies (uncertainties below 10^{-2}) for experimental uncertainties with standard deviation smaller than 10^{-3}. Starting from 80, the number of rays measured had no considerable influence on the accuracy of the GRIN's reconstruction. Using SIMPLEX and quasi-Newton algorithms comparable results are obtained, but the latter is more efficient.

KEYWORDS: GRIN lens, GRIN reconstruction, optimization problem, ray tracer, human eye

MSC: 65K10, 34K28, 78-04

RESUMEN
En este trabajo se presenta un método para la reconstrucción del GRIN (Gradient Index) en cristalinos, basado en la obtención de los parámetros que lo caracterizan a partir de mediciones de la trayectoria de un rayo de luz antes y después de atravesar el cristalino. El problema a resolver se plantea como un problema de optimización. Se describe el procedimiento para calcular la trayectoria de un rayo de luz a través del cristalino usando el principio de Fermat, como una parte fundamental del método. Para resolver el problema de optimización se usaron los algoritmos SIMPLEX (Nelder & Mead) y quasi-Newton (BFGS), combinados con el de búsqueda global BASIN-HOPPING. Para evaluar el método, se usó el un modelo de GRIN parabólico. Los resultados muestran que es posible reconstruir el GRIN con buena exactitud (errores menores a 10^{-2}) para errores en las mediciones de la trayectoria del rayo con desviación estándar menor que 10^{-3}. A partir de 80, el número de mediciones no influye significativamente en la reconstrucción del GRIN. Los resultados obtenidos con los métodos SIMPLEX y quasi-Newton son similares, siendo el este último el más eficiente.

1. INTRODUCTION

The mechanism of the human vision is a complex process that involves the coordinated work of the cornea, lens and muscles that control the amount of light entering the eye and its route, to form an in-focus image on the retina. In this process, the crystalline plays a key role. Firstly, the crystalline is responsible for focusing on objects at different distances through the variation of its curvature and thickness in a process known as accommodation. Secondly, the crystalline has a non-uniform distribution of refractive index, conforming a GRIN (Gradient Index) lens [1] which causes the light to follow a complex path inside it and consequently, an enhanced optical power. The ray trajectory inside the crystalline is usually calculated solving a system of ordinary differential equations.

To perform a realistic optical modeling of a human eye, it is necessary to include accurate information of the distribution of refraction index in the crystalline. The way to get this information is based on estimates of the refractive indexes in real crystallines. This is a complex task because the GRIN estimate *in vivo* is only possible using Magnetic Resonance Imaging (MRI) [12-14]. It is also possible to make direct measurements *in vitro* using Abbe refractometers [23], but in this case the crystalline is destroyed, which adds uncertainties in the results. Other methods frequently used are optimization methods based on the experimental measurements of ray paths traversing the crystalline.

[1] roberto.cruz.rdg@gmail.com

In these methods, the idea is to assume some model for the GRIN and then relate mathematically (through an optimization problem) the measurements of ray paths with the parameters of the model. The measurements of the ray path are obtained from *in vitro* experiments [4, 9, 14] or from images of Optical Coherence Tomography (OCT) [5, 6]. In [9] the GRIN of fish Black Oreo Dory crystalline is reconstructed using the measurements of ray paths obtained from OCT images. The results are compared with those obtained using MRI techniques. Experimental data obtained from OCT images are also used in [6] to reconstruct the human crystalline GRIN. In Refs. [5, 14] an optimization method based on the ray paths is used to study the change in human crystalline GRIN with the age and accommodation respectively. In Ref. [4] the accuracy of optimization methods for reconstruction of GRIN of the crystalline lens was evaluated, using two sources of input data: first, direction cosines of deflected rays and ray impacts obtained from ray tracing and second, optical path differences from Optical Coherence Tomography (OCT). They used the GRIN models proposed in [10, 11].
In the above references the corresponding optimization problems were solved with standard search algorithms. In [9], they used a routine based on *conjugate directions* method [21]. In [4, 5, 6], they used a mixture of a global genetic algorithm (see [6] for details) and the local search Nelder & Mead's SIMPLEX [18].
This paper presents a simple yet precise method to obtain the parameters of the GRIN from experimental measurements of the ray path before and after traversing the crystalline lens. These measurements can be obtained from a simple experiment as described in [14]. The method is based on assuming a mathematical model for the GRIN lens that depends on certain parameters and then using an optimization algorithm to find the values of these parameters which best reproduce experimental measurements. Since the proposed method depends directly on the calculation of the ray paths through the crystalline, this procedure is described in detail. The paper is divided as follows: In section 2 we develop the optimization model to be used and present the numerical experiments. In section 3 we present our results and discuss them. Finally, some conclusions are drawn.

2. THE MODEL

2.1. A grin model for the crystalline

The crystalline is a convex lens limited by two aspherical surfaces [2] (see Fig. 1) which could be mathematically described by the equation:

$$S(z,w) = w^2 + (1+Q)z^2 - 2Rz = 0, \tag{1}$$

where $w^2 = x^2 + y^2$, Q is the asphericity parameter and R is the radius in the vicinity of the optical axis.

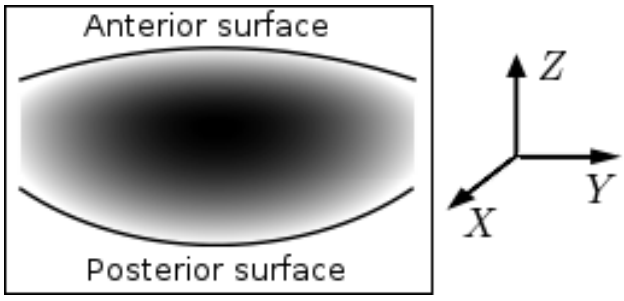

Figure 1 A Human Crystalline, right side the coordinate system used in the model.

Assuming radial symmetry, the distribution of refraction indexes of the GRIN lens can be described by a bi-parabolic model [3]. Dividing the crystalline in two with a plane perpendicular to the optical axis and passing through the nucleus, (which is located on the optical axis, at a distance d from the vertex of the anterior surface), the index distribution in this model could be written as follows:

$$n(z,w) = \begin{cases} n_N + n_a(z-d)^2 + n_0 w^2, & z < d \\ n_N + n_p(z-d)^2 + n_0 w^2, & z \le d. \end{cases} \tag{2}$$

The function $n(z,w)$ represents the refractive index at each point inside the crystalline. The form of Eq. (2) is valid when the coordinate origin is in the vertex of the anterior surface. So, the distribution of refraction indexes inside the crystalline is determined by the following parameters:

- n_N : Refractive index in the crystalline nucleus. The maximum refractive index inside the crystalline.
- d: The distance from the vertex to the nucleus.
- n_a: Parameter representing the axial variation of the GRIN in the anterior portion.
- n_p : Parameter representing the axial variation of the GRIN in the posterior portion.
- n_0 : Parameter representing the radial variation of the GRIN.

Our task is to determine the values of the above parameters from experimental measurements of deviations of ray paths when traveling through the crystalline.

2.2. Formulation of the analytical problem

Let us suppose that from a certain experiment (as proposed, for instance, in [14], see Fig. 2(a)), we measure the incoming and exit points and directions of a light ray in an isolated crystalline. For this, a transparent liquid (with refraction index equal to that of the aqueous humor) fills a chamber where the crystalline to be tested is located. To see the ray paths, a few drops of milk are added to the liquid.

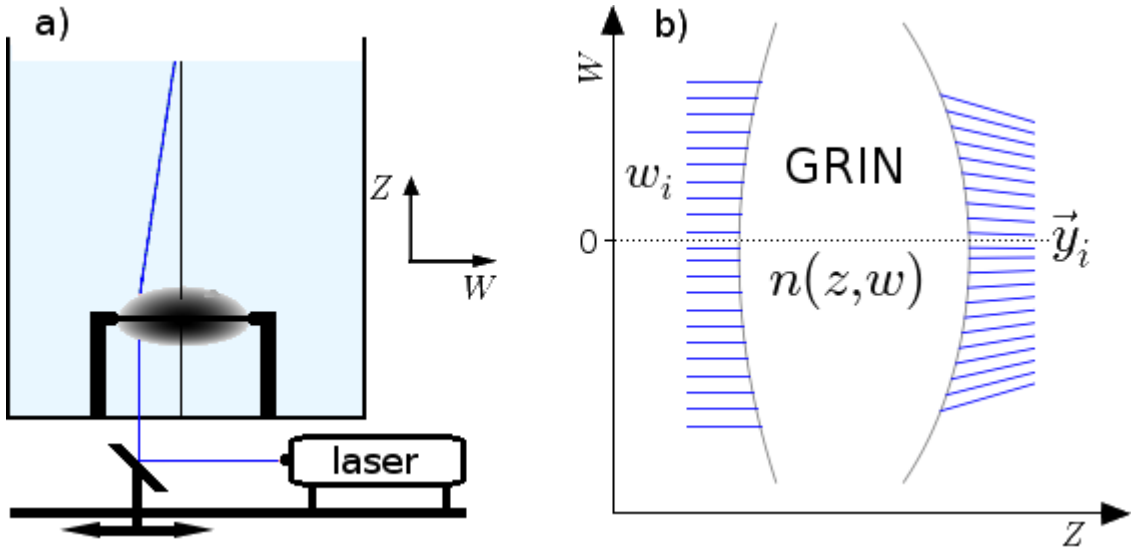

Figure 2 The experimental procedure. (a) Sketch of an experiment to measure ray paths before and after passing through the crystalline. (b) A set of measurements.

As can be seen in figure 2(a), the incoming ray is parallel to the Z axis. The experiment consists in measuring the values $w_i \in R$ and $\vec{y}_i \in R^2$ with $i = 1,2,\dots,N$ (see Fig. 2(b)). The values w_i indicate the distance to the Z axis (central axis of the lens) where the incoming ray enters on the anterior crystalline surface. The vectors $\vec{y}_i$ have the form (w_{ei}, m_i), being w_{ei} the distance from Z axis to the point where the outgoing ray exits the crystalline on its posterior surface and m_i the slope of this ray in the ZW plane.

Let us suppose further that we have a function $\vec{F}: R \to R^2$, depending on M parameters $c_1, c_2, c_3, \dots, c_M$:

$$\vec{y} = \vec{F}(w|c_1, c_2, c_3, \dots, c_M), \tag{3}$$

representing the law that associates each value of w (incoming ray) with the corresponding vector $\vec{y} = (w_e, m)$ (exit ray). Evidently, evaluating $\vec{F}(w|c_1, c_2, c_3, \dots, c_M)$ involves calculating the path of a ray of light through the crystalline lens.

Because the light path through the crystalline is determined by the GRIN, we can assume that the above parameters $c_1, c_2, \dots, c_M$ characterize the GRIN. That said, we addressed the problem of reconstructing the GRIN as an optimization one: determining values $c_1, c_2, \dots, c_M$ for the $\vec{y} = \vec{F}(w|c_1, c_2, \dots, c_M)$ best fit to the experimental data, that is:

$$\min_{c_1,c_2,..,c_M} \sum_{i=1}^{N} \left\| \vec{y}_i - \vec{F}\left(w_i | c_1, c_{2,} c_{3,} \dots, c_M\right) \right\|_2^2 . \quad (4)$$

In the case of a GRIN biparabolic model, the parameters $c_1, c_2, \dots, c_M$ are replaced by n_N, d, n_a, n_p and n_0. To solve the above problem by applying some optimization algorithm, it is necessary numerically evaluate the function $\vec{F}(w|c_1, c_2, c_3, \dots, c_M)$. As discussed below, the calculation of the ray path through the lens involves the numerical solution of a system of differential equations. Therefore, it is necessary to have a way to efficiently evaluate the function $\vec{F}\left(w|c_{1,} c_2, c_3, \dots, c_M\right)$, because each evaluation of the merit function (4) leads to N evaluations of $\vec{F}\left(w|c_{1,} c_{2,} c_3, \dots, c_M\right)$.

2.3. Calculation the ray path through the crystalline

The calculation of the light path through the crystalline lens (evaluation of $\vec{F}\left(w|c_{1,} c_2, c_3, \dots, c_M\right)$) has four steps (see Fig. 3):

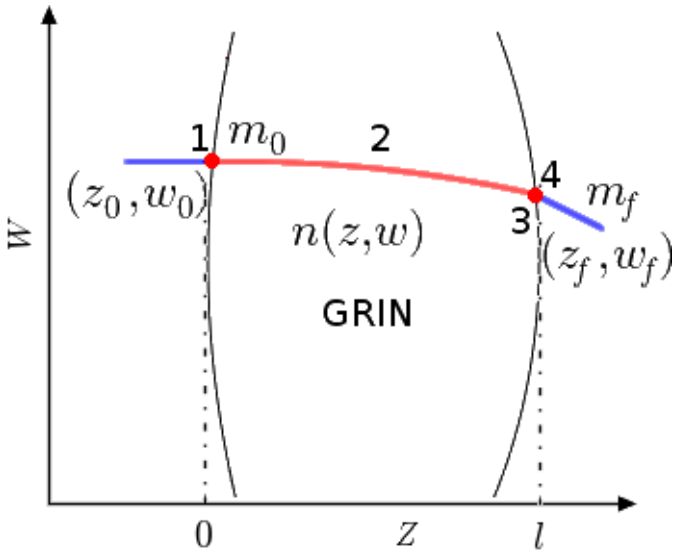

Figure 3 Schematic path of a light ray through the crystalline, showing the parameters to be measured and the steps of the calculation procedure.

1) Application of Snell's law at the point (z_0, w_0) where the incoming ray touches the anterior surface of the crystalline.
2) Calculation of the light path through the crystalline. This is the most complex and computationally expensive step.
3) Determination of the exit point (z_f, w_f) on the posterior crystalline surface.
4) Application of Snell's law in the exit point (z_f, w_f).

The Snell´s law state that the ratio of the sines of the angles of incidence and refraction of a light rays in a surface is equal to the reciprocal of the ratio of the indices of refraction. So, the steps one and four are calculated using a simple formula:

$$\sin(\alpha_2) = \frac{n_1 \sin(\alpha_1)}{n_2},$$

where α_1 y α_2 are incident and refraction ray angles (relate with the ray slopes values) and n_1 and n_2 the refraction indexes (e.g.: the refraction indexes in the liquid and at the point (z_0, w_0)).
The second step is the most time consuming, because it depends on the numerical solution of a system of differential equations. Below is how we proceeded.
To calculate the path of light through the GRIN lens, Fermat's principle (as used in Ref. [17]) was applied. This is a variational principle which states [1] that the path followed by the light to propagate from one point P_1 to

another point P_2, in a medium of refraction index n, is such that the optical path (or the time) spent in traveling is an extremal relative to possible variations of the trajectory.
The optical path is represented by the functional:

$$H[\vec{r}(s)] = \int_{P_1}^{P_2} n[\vec{r}(s)]ds, \tag{5}$$

where s is the path and $n[\vec{r}\,(s)]$ the refractive index of the medium in each point of the trajectory. The extremal condition is expressed as:

$$\delta H[\vec{r}(s)] = \delta \int_{P_1}^{P_2} n[\vec{r}(s)]ds = 0\,. \tag{6}$$

For the crystalline lens, choosing the reference frame ZW, equation (6) becomes:

$$\delta \int_{P_1}^{P_2} n\big(z, w(z)\big)\sqrt{1 + w'(z)}dz = 0\,. \tag{7}$$

Here, $w(z)$ describes the trajectory of the extremal (in this case minimal) optical path. Function $w(z)$ is determined by solving the corresponding Euler-Lagrange equation associated with (7):

$$\frac{\partial L}{\partial w} - \frac{d}{dz}\left[\frac{\partial L}{\partial w'}\right] = 0, \tag{8}$$

where $L(z, w, w') = n(z, w)\sqrt{1 + w'^2}$.

Writing equation (8) for $w(z)$ the following differential equation is obtained:

$$w''(z) = \frac{1}{n(z,w)}(1 + w'^2)\left(\frac{\partial n}{\partial w} - \frac{\partial n}{\partial z}w'\right). \tag{9}$$

To solve the above equation, it is necessary to add the initial conditions:

$$w(z_0) = w_0 \tag{10}$$

$$w'(z_0) = m_0, \tag{11}$$

where (z_0, w_0) is the entry point of the incoming ray and m_0 its slope.
To solve the Cauchy problem defined by equations (9), (10) and (11) we used the integration scheme Runge-Kutta-45 (RK45) described in [7, 20].
The exit point at the posterior crystalline surface (the intercept of the curve $w(z)$ with the posterior surface), was calculated with the following procedure.
Equation (9) was integrated numerically using RK45 up to the first point (z_n, w_n) found at the right of the posterior surface, so it was needed to test if the ray path was inside the crystalline at each iteration step. After that, we used a Hermite polynomial of degree four $P_4(z)$ as described in [20, 8], to approximate $w(z)$ between points (z_{n-1}, w_{n-1}) and (z_n, w_n). Finally, we used a bisection method to find the interception point (z_f, w_f) between $P_4(z)$ and the posterior crystalline surface:

$$S_p(z, w) = w^2 + \big(1 + Q_p\big)(z - l)^2 - 2R_p(z - l) = 0, \tag{12}$$

where l is the axial length of the lens and Q_p and R_p are asphericity parameter and radius of the posterior surface, respectively.

In our previous work [3] different methods for the calculation of the ray paths through the GRIN lens were compared. The procedure described above results the most efficient computationally.

2.4. Optimization procedure

To solve the optimization problem (4), two algorithms are used. The first is the Nelder & Mead's SIMPLEX algorithm [18]. This has the advantage of requiring only function evaluations, but is less efficient than other methods that use gradient information. The second is the algorithm of quasi-Newton BFGS [19]. This is usually more efficient than SIMPLEX algorithm, but requires calculating the gradient of the merit function, which in this case is obtained approximately using finite differences.

2.5. Numerical experiments

To validate our proposal for the GRIN reconstruction, firstly, we tested if the procedures could reconstruct a constant index lens, *i. e.* only $n_N \neq 0$ (see Eq. (2)). For doing this, the following procedure was applied:

1) Generating numerical data
   - The program selects a distance w_i and send the ray parallel to the optic axis impinging the anterior surface. Using Snell's law and rectilinear path ($n = const$) calculates w_{ei} and m_i.
   - A Gaussian noise with zero mean and standard deviation σ is added to w_{ei} and m_i.
   - Repeat previous points 100 times for different distances. These values are the data of one experiment.
   - Repeat previous points 50 times, this is the number of experiments.
2) Reconstructing the GRIN parameters
   - For each group of 100 measurements (w_{ei} , m_i) the optimization algorithm is applied, obtaining the parameters of the reconstructed GRIN.
   - In N selected points inside the crystalline the value of the difference of the reconstructed value of the refraction index and the real value is calculated, and the root mean square value (RMS) is the uncertainty of this particular reconstruction.
   - The global uncertainty is the average value calculated for the 50 reconstructions.

This reconstruction serves us as a first indication of the quality of the numerical tools developed. Then the same procedure is repeated but with values of the GRIN obtained from [17].
For the numerical experiments, the ray tracer algorithm was implemented in C++ and the optimization problems were solved in Python using the functions **fmin()**, **fmin_bfgs()** and **basinhopping()**, from the Scipy module [15].

3. RESULTS AND DISCUSSION

To evaluate the GRIN reconstruction method, several experimental measurements of the light path were simulated. The nominal GRIN parameters and anatomical parameters that were used in the simulations are shown in Tables 1 and 2. These data were taken from [17] and agree with the average values reported for a young human crystalline.

n_N	d (mm)	n_a	n_p	n_0
1.407	1.59	-0.015427	-0.006605	-0.001978

Table 1 Nominal GRIN parameters

Surface	Q	R (mm)	Position (mm)
Anterior	-0.94	12.4	0.0
Posterior	0.96	-8.1	4.02

Table 2 Parameters of the surfaces of the GRIN lens.

Experimental measurements (w_i , $\vec{y}_i$) were simulated computationally using the algorithm described above. For each GRIN reconstruction, 100 measurements (w_i , $\vec{y}_i$) were simulated varying w_i uniformly in the range

$[-3, 3]$ mm. To explore the limits of the method, we have assumed Gaussian errors in the components of vector $\vec{y}_i$ with standard deviation $\sigma = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$ and 10^{-6}, to consider various levels of experimental uncertainties.

In carrying out the experiments it was observed that the problem (4) has several local minima, so it was decided to combine the methods BFGS and SIMPLEX with the global search algorithm BASIN-HOOPING [22]. These combinations do not guarantee to obtain the global minima, but the results are better than those obtained using only local algorithms SIMPLEX or BFGS. BASIN-HOOPING is a stochastic algorithm which attempts to find the global minimum of a smooth scalar function of one or more variables. The algorithm is iterative and in each iteration a local search is used for minimization, starting from points generated randomly. In our case, the number of iterations of BASIN-HOOPING was set to 10, with more iterations was not observed a significant improvement of the results.

Table 3 shows restrictions boxes used in the numerical experiments. Limit restrictions were chosen such that it was impossible to obtain a physically unrealistic (e.g.: indexes of refraction less than 1) or physiologically incorrect (e.g.: refractive index values far from the average reports) GRIN.

d (mm)	n_N	n_a	n_p	n_0
[1,2]	$[1.38, 1.42]$	$[-0.38, 0]$	$[-0.09, 0]$	$[-0.42, 0]$

Table 3 Box restrictions for GRIN parameters.

3.1. Accuracy in the grin reconstruction

To evaluate the accuracy of the reconstruction, we selected a regular grid of points with steps 10^{-2} mm inside the crystalline [4], located at distances $|w| \leq 3$ mm from the optical axis. In the grid points the nominal and reconstructed refraction index values are calculated. Two measures of the uncertainty are used: the root mean square (RMS) of the deviation of the reconstructed values relative to the nominal and the maximum value of their differences. The nominal values are $n = 1.42$ for the constant index lens and, for the GRIN, the values recovered when substituting in Eq. (2) the values of the constants (for instance, those of Table 1).

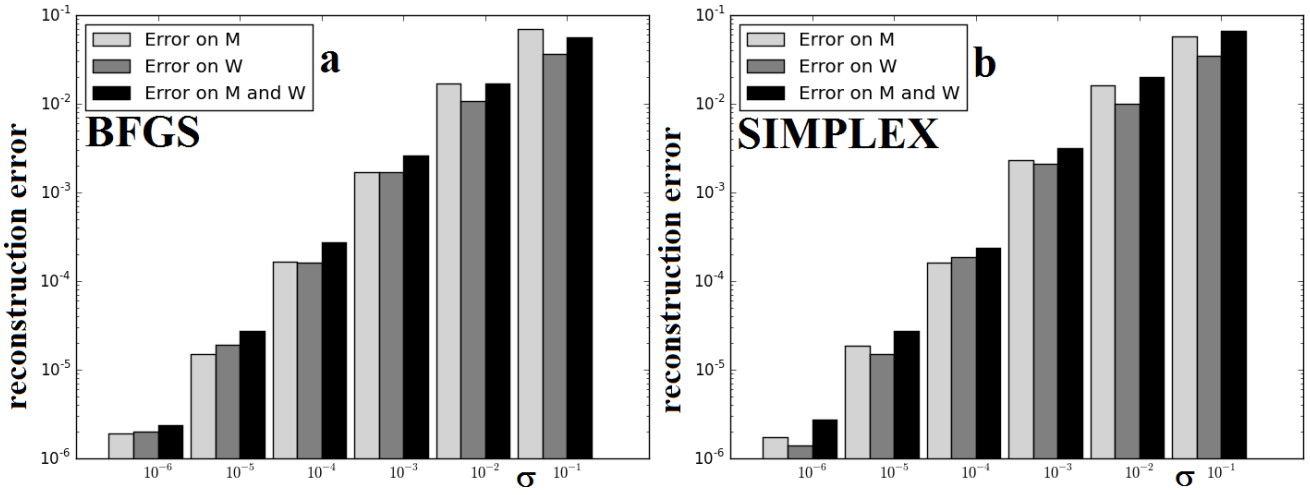

Figure 4 Errors in the GRIN reconstruction depending on errors in vectors $\vec{y}_i$ for the crystalline with constant n = 1.42 (a) using BFGS, (b) using SIMPLEX. For each σ three bars are shown: light gray (left) considers only errors in the slope of the outgoing ray, gray (center) only considers the error in the position of the exit point, black (right) errors in both magnitudes.

Figure 4 shows the calculated RMS errors in the GRIN reconstruction depending on the error in measurements of vectors $\vec{y}_i$ in the case that the lens has a constant refraction index $n = 1.42$. The data shown in Fig. 4 are average errors of 50 GRIN reconstructions for each value of the standard deviation. Three values are shown for each σ: considering only errors in m_i, considering only errors in w_{ei} and considering errors in both magnitudes. The results when the maximum error is considered are consistent with those shown in Fig. 4.

As can be seen, the error in the reconstruction of the GRIN decreases with decreasing errors in the input data. The results do not differ significantly by using the BFGS method (Fig. 4(a)) or SIMPLEX method (Fig. 4(b)). In addition, it was found that the time spent on a reconstruction of the GRIN using BASIN-HOPPING-BFGS is much lower than using BASIN-HOPPING-SIMPLEX. It was also noted that, for the two largest Gaussian errors ($\sigma = 10^{-2}$ and $= 10^{-1}$), the reconstructed parameters of model, in general, take the extreme values of the restriction boxes. This phenomenon suggest that the global minima cannot be found inside the region defined by Table 3, so $\sigma = 10^{-3}$ can be taken as the maximum permissible error, in this case the error in the reconstruction of GRIN is less than 10^{-2} . It is also possible to note that the resulting error increases almost linearly with the measurement error.

From Fig. 4 another interesting feature appears: the error in the determination of the slope influences the result in greater extent than the error in w_{ei} . That is good, because for obvious reasons it is easier to measure with good accuracy the ray's slope than the position of the exit point.

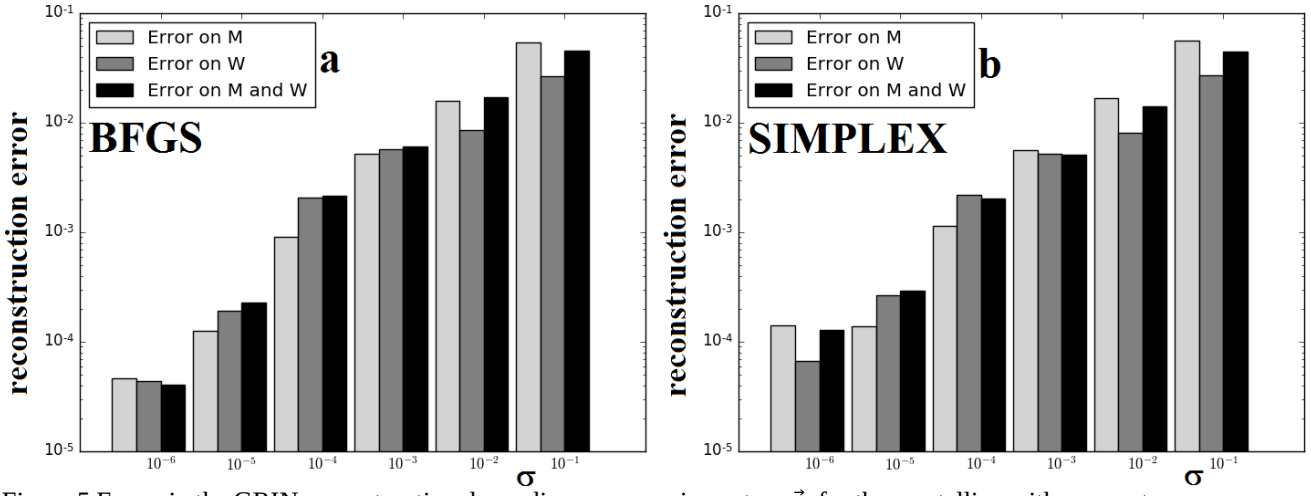

Figure 5 Errors in the GRIN reconstruction depending on errors in vectors $\vec{y}_i$ for the crystalline with parameters taken from Table 1 (a) using BFGS, (b) using SIMPLEX. For each σ three bars are shown: light gray (left) considers only errors in the slope of the outgoing ray, gray (center) only considers the error in the position of the exit point, black (right) errors in both magnitudes.

Figure 5 is like Fig. 4, only the values are calculated for a GRIN with parameters shown in Table 1. In this case the error of reconstruction is influenced by both magnitudes, though for the larger σ the error in the slope is dominant. There are two important differences between Figs. 4 and 5. Firstly the errors in the reconstruction of the real GRIN are considerably larger for the smaller values of σ (one order of magnitude). Secondly the increase of σ influences differently in both cases: while for the constant index the reconstruction errors increase linearly, for the GRIN there are easily seen two stages, in the first -for low σ- the errors increase linearly, but then there is a change of slope, and for the larger σ the reconstruction errors for both cases are of the same magnitude.

For both cases, constant index and GRIN index, the reconstruction errors (RMS) were bellow 0.6×10^{-2} , except for experiment with $\sigma = 10^{-2}$ and $\sigma = 10^{-1}$. This result is comparable with Refs. [6, 4]. In the first [6], they obtain reconstruction errors of 0.4×10^{-2}, using an optimization algorithm based on OCT images and the four variables GRIN model [16]. In the second study [4], they found reconstruction errors bellow 0.5×10^{-2} for two variable Goncharov model [10, 11] and higher errors for three and four variables Goncharov models. The main differences with the method presented here are the input data, the formula for ray tracing and the consideration of the experimental error on the surface parameters of the crystalline. Though the errors in the reconstruction for $\sigma > 10^{-3}$ are not very good, it is not an obstacle in the use of the proposed method, as it is possible to obtain errors of around 10^{-3} in the experimental determination of the direction cosines of light rays [4]. Such errors are easily attainable when determining the ray slope, because in both methods the magnitudes are calculated fitting a first order polynomial to the ray path in an image.

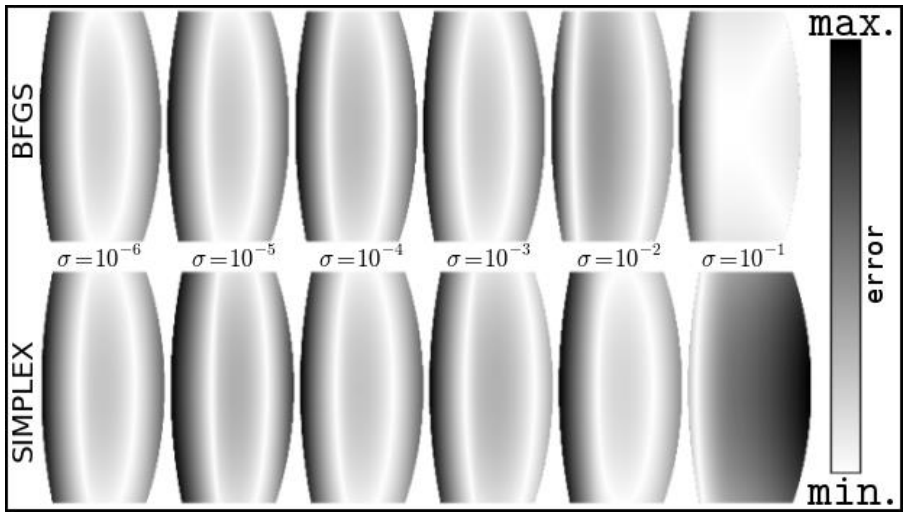

Figure 6 Spatial distribution of the absolute error in the GRIN reconstruction inside the crystalline. The scale shows relative values, see text.

Figure 6 shows the spatial distribution of the absolute error in one of the GRIN reconstructions inside the crystalline for each Gaussian error level. In the figure, gray scale is relative, showing the spatial distribution of the errors and not their values.

As shown in Figure 6, the larger errors are found near the lens surfaces. This is very good because in practice it is very difficult to estimate the refractive index inside the lens, whereas in areas near the surface it is possible to make direct measurements with Abbe refractometers. The poor reconstruction in the surface was also reported in [6], but only for older human lenses. The possible cause is the larger role of the nucleus in the description of the lens GRIN in old lenses, in comparison to young lenses [6]. In [4], they also note that the error increased towards the surface for most of the reconstruction experiments. To have a quantitative description, we calculate the reconstructing error on the nucleus and surface only, the results are shown in Fig. 7.

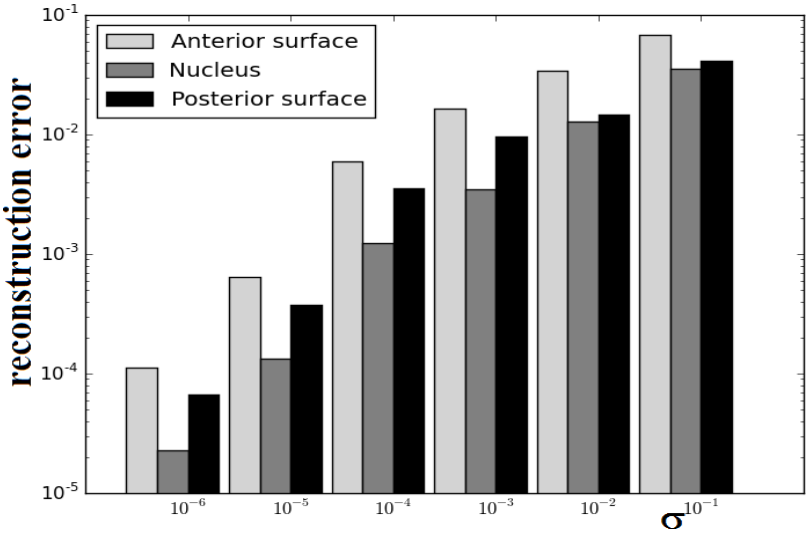

Figure 7 Reconstruction errors for surfaces and nucleus only. For each σ three bars are shown: light gray (left) represent error in anterior surface, gray (center) error in nucleus and black (right) error in posterior surface.

As can be seen in Fig. 7, the errors in both surfaces are higher than errors in nucleus for all cases, especially on the anterior surface. For $\sigma = 10^{-6}, \ldots, 10^{-3}$ the errors in the surfaces (average of them) are about three times the nucleus errors. We obtain reconstruction errors in surface and nucleus below 1.3×10^{-2} and 3.5×10^{-3} respectively.

3.2. Influence of the number of experimental measurements on the grin reconstruction

To determine how the number of experimental measurements influences the GRIN reconstruction, several reconstructions were performed considering different number of measurements for building the merit function (4). For errors $\sigma = 10^{-6}$ to $\sigma = 10^{-3}$ were performed 50 reconstructions varying the number of measurements from 20 to 200, then we calculated reconstruction error (as the average of 50 reconstructions) for every number of measurements. We also calculated the mean execution time of one GRIN reconstruction.
For all errors, it was observed that with increasing number of measurements increases the accuracy of the GRIN reconstruction, but starting from 80, the number of experimental measurements does not significantly improve the accuracy of the GRIN reconstruction. The mean execution time of one reconstruction grows linearly with the number of measurements, in our case (Intel(R) Celeron(R) CPU E3400 @ 2.60GHz) one reconstruction with 80 measurements takes about 50 seconds.

4. CONCLUSIONS

In this contribution, we describe a method for the reconstruction of human crystalline GRIN, from measurements of the ray path before and after traversing the crystalline lens. The problem of GRIN reconstruction was addressed as an optimization one. To evaluate the method several numerical experiments were performed, yielding the following conclusions:

- The existence of several local minima of the merit function, made it necessary to combine the global search algorithm BASIN-HOPPING with local search algorithm BGFS and SIMPLEX.
- The results obtained with the SIMPLEX and BGFS algorithms are similar. However, the BFGS method is more efficient, so we recommend BFGS algorithm for practical GRIN reconstructions.
- For Gaussian errors on experimental measurement of the ray path small enough (with standard deviation less or equal than 10^{-3}) the GRIN can be reconstructed with good accuracy.
- The accuracy of the GRIN reconstruction is better in crystalline nucleus than in the surface, the larger errors corresponding to the anterior surface.
- It was determined that 80 measurements of ray path are sufficient for obtaining a good GRIN reconstruction. Furthermore, the reconstruction time increases linearly with the increase of the number of measurements.

Of course, this method could be potentially applied to any GRIN lens.

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