AN IMPROVED NEWTON-RAPHSON METHOD FOR SOLVING TWO SIMULTANEOUS NONLINEAR EQUATIONS

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ABSTRACT
In this note, an improved Newton-Raphson (INR) method based on the classical Newton-Raphson (N-R) method is proposed for solving simultaneous nonlinear equations (SNEs). An implicit function theorem and a resulting modified Newton Raphson method for roots of the functions between finite dimensional spaces is being analyzed without assuming non-singularity of the Jacobian at the initial approximation. The proposed method has faster convergence rate with respect to the other modified Newton's method for solving SNEs.

KEYWORDS: Newton-Raphson method; Improved Newton-Raphson method; Simultaneous Nonlinear equations.

MSC: 65H10.

RESUMEN
En esta nota proponemos un método de Newton-Raphson mejorado (INR) basado en el clásico método de Newton-Raphson (N-R) para resolver ecuaciones no lineales simultaneas (SNEs). Un teorema de la función implícita y el resultante método de Newton Raphson mejorado para las raíces de las funciones entre espacios de dimensión finita ha sido analizado sin asumir la no-singularidad del Jacobiano en la inicial aproximación. El propuesto método tiene una razón de convergencia mas rápida con respecto a los otros métodos modificado de Newton para resolver SNEs.

1. INTRODUCTION

Many practical problems demand for a solution of a set of simultaneous nonlinear equations. The problems will vary greatly from one discipline to another, but the basic mathematical formulation remains the same (for details see Refs. [8,9,11]).
The N-R method for solving an equation

$$f(x) = 0 \tag{1}$$

is based upon the convergence, under the suitable conditions of the sequence

$$x_{p+1} = x_p - \frac{f(x_p)}{f'(x_p)}, \quad p = 0,1,2, \ldots\ldots \tag{2}$$

to the solution of (1), where x_0 is an initial approximation to that solution. The modified N-R method uses, instead of (2), the sequence

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$$x_{p+1} = x_p - \frac{f(x_p)}{f'(x_0)}, \quad p = 0,1,2 ,...... \tag{3}$$

These are described in detail in [4, 7]. Extensions of system of equations

$$\begin{aligned} & f_1(x_1, x_2, ..., x_n) = 0 \\ & ... \\ & .... \\ & f_m(x_1, x_2, ..., x_n) = 0 \end{aligned} \tag{4}$$

are instant in case: $m = n$ (see the Ref. [4]). The analogs of (2) and (3) are respectively

$$x_{p+1} = x_p - (J(x_p))^{-1} f(x_p), \quad p = 0,1,2 ,...... \tag{5}$$

$$x_{p+1} = x_p - (J(x_0))^{-1} f(x_p), \quad p = 0,1,2 ,...... \tag{6}$$

where x is the vector with components x_j, $j = 1,2,...,n$, $f(y)$ is the vector with components $f_i(x)$, $i = 1,2,...,n$, $J(x)$ is the Jacobian matrix, whose (i, j)th element is $\frac{\partial f_i(x)}{\partial x_j}$ and x_0 is an initial approximation to the solution (4). When $m \neq n$, the composite N-R gradient method of Hart and Motzkin [5], may also be applicable, provided the rank of the Jacobian matrix is n at the solution.

The idea of the proposed method is similar to a two stage method. The classical Newton Raphson method will play as of a predictor to obtain the approximate value of one variable and the proposed algorithm would be as a corrector of the other variable. Solution of a system of nonlinear equations, by modifying Newton's method and the semi-implicit root solver methods have been done by Broyden [2], Hardaway [3], Scheffel and Håkansson [9]. Long date back, in 1965, Ben-Israel [1] generalized the classical Newton Raphson method for finding the solutions of system of equations. Recently, a new method for the numerical solution of simultaneous nonlinear equation has been discussed by Ibidapo-Obe et al [6]. Shaw and Mukhopadhyay [10] have already successfully introduced an improved idea on classical Regula-Falsi method for solving nonlinear equations.

In this paper, a modified form of Newton Raphson method for solving SNEs is proposed with some numerical discussions. The convergence of the proposed scheme is ensured. A comparison study is being made with the Newton's method.

2. DERIVATION OF THE NEW IMPROVED METHOD

Let a system of two nonlinear equations with two unknowns be given as

$$f(x, y) = 0 \tag{7}$$

$$g(x, y) = 0 \tag{8}$$

The classical Newton Raphson method is aptly called the method of tangent since the method is based on the existence of derivatives of the function. To solve the equations (7) and (8) we assume that $f(x, y)$ and $g(x, y)$ are real continuous functions together with its first and second order partial derivatives lying within the interval where the solution of the equations is supposed to lie.

Suppose that (x_0, y_0) is an initial approximation of the solution, say (ξ, η), of the nonlinear equations (7) and (8).
Now equation of the tangent of the equation (7) at the point where the corresponding abscissa is at $x = x_0$ is

$$\left(y - y_0^{(f)}\right) f_x\left(x_0, y_0^{(f)}\right) = (x - x_0) f_y\left(x_0, y_0^{(f)}\right) \tag{9}$$

Again, equation of the tangent of the equation (8) at the point where the corresponding abscissa is at $x = x_0$ is

$$\left(y - y_0^{(g)}\right) g_x\left(x_0, y_0^{(g)}\right) = (x - x_0) g_y\left(x_0, y_0^{(g)}\right) \tag{10}$$

where $y_0^{(f)}$ and $y_0^{(g)}$ are corresponding (may be approximated) values of $f(x, y)$ and $g(x, y)$ at $x = x_0$ respectively.
Geometrically, $y_0^{(f)}$ and $y_0^{(g)}$ are the point of intersections of the functions $f(x, y)$ and $g(x, y)$ respectively to the straight line $x = x_0$. Therefore, to obtain $y_0^{(f)}$ we can apply classical Newton-Raphson method to the nonlinear equation $f(x_0, y) = 0$ with the initial approximation $y = y_0$. Thus, By N-R method,

$$y_{0,m+1}^{(f)} = y_{0,m}^{(f)} + \frac{x_n - f\left(x_0, y_{0,m}^{(f)}\right)}{f_y\left(x_0, y_{0,m}^{(f)}\right)}, \; m = 0, 1, 2, \; ....$$

with

$$y_{0,0}^{(f)} = y_0.$$

Similarly,

$$y_{0,m+1}^{(g)} = y_{0,m}^{(g)} + \frac{x_n - g\left(x_0, y_{0,m}^{(g)}\right)}{g_y\left(x_0, y_{0,m}^{(g)}\right)}, \; m = 0, 1, 2, \; ....$$

with $y_{0,0}^{(g)} = y_0$, and up to a desire degree of accuracy, ε say, when

$$\left| y_{0,m+1}^{(\,)} - y_{0,m}^{(\,)} \right| < \varepsilon$$

we consider

$$y_{0,m}^{(\,)} = y_0^{(\,)}.$$

Solving equations (9) and (10) we get,

$$x = x_0 - \left(y_0^{(f)} - y_0^{(g)}\right)\left[\frac{f_x g_x}{f_x g_y - f_y g_x}\right]_{\text{at } x = x_0} = x_1 \text{ (say)}$$

which gives the 1st iteration for the solution, and the successive approximations are given by

$$x_1, x_2, x_3, ..., \; x_n, x_{n+1}, ...$$

where

$$x_{n+1} = x_n - \left(y_n^{(f)} - y_n^{(g)}\right)\left[\frac{f_x g_x}{f_x g_y - f_y g_x}\right]_{\text{at } x = x_n} \tag{11}$$

In which $y_n^{(f)}$ and $y_n^{(g)}$ may be determined by classical Newton-Raphson method, up to a desire order of precision, as

$$y_{n,m+1}^{(f)} = y_{n,m}^{(f)} + \frac{x_n - f\left(x_n, y_{n,m}^{(f)}\right)}{f_y\left(x_n, y_{n,m}^{(f)}\right)}, \quad m, n = 0,1,2, \; .... \tag{12}$$

with

$$y_{0,0}^{(f)} = y_0$$

and

$$y_{n,m+1}^{(g)} = y_{n,m}^{(g)} + \frac{x_n - g\left(x_n, y_{n,m}^{(g)}\right)}{g_y\left(x_n, y_{n,m}^{(g)}\right)}, \quad m, n = 0,1,2, \; .... \tag{13}$$

with $y_{0,0}^{(g)} = y_0$.

2.1 Geometrical explanation of the new improved method

In this figure, (x_0, y_0) is the initial guess of the solution, now if we draw one straight line parallel to y-axis then that straight line intersects two curves at least two points. To determine the approximate values of those two points we use classical Newton-Raphson method. Then we draw to tangents at those two points $A\left(x_0, y_0^{(f)}\right)$ and $B\left(x_0, y_0^{(g)}\right)$, say. These two tangents intersects at $C(x_1, y_1)$, say, that will be considered as the next iterative value of the solution and so on.

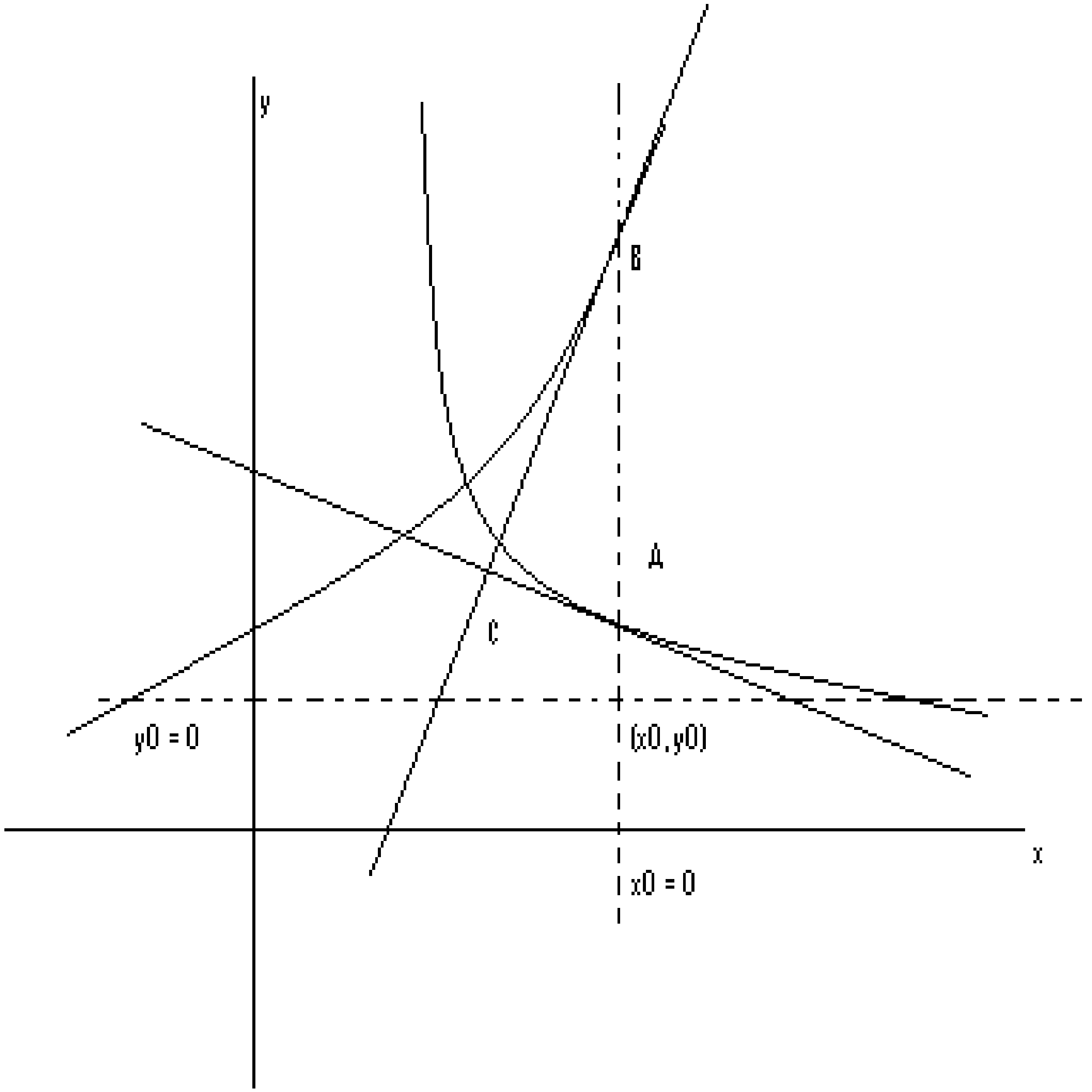

Figure 1: Geometry for improved N-R method

2.2 Algorithm for the INR method

Step 1: Get x_0, y_0 (initial approximation) and ε (precision).
Step 2: Compute

$$y^{(f)}_{n,m+1} = y^{(f)}_{n,m} + \frac{x_n - f\left(x_n, y^{(f)}_{n,m}\right)}{f_y\left(x_n, y^{(f)}_{n,m}\right)} \text{ and } y^{(g)}_{p,q+1} = y^{(g)}_{p,q} + \frac{x_p - g\left(x_p, y^{(g)}_{p,q}\right)}{g_y\left(x_p, y^{(g)}_{p,q}\right)}.$$

Step 3: If $\left|y^{(f)}_{n,m+1} - y^{(f)}_{n,m}\right| < \varepsilon$ and $\left|y^{(g)}_{p,q+1} - y^{(g)}_{p,q}\right| < \varepsilon$

then $y_{n,m}^{(f)} = y_n^{(f)}$, $y_{p,q}^{(g)} = y_p^{(g)}$ and compute

$$x_{n+1} = x_n - \left(y_n^{(f)} - y_n^{(g)}\right)\left[\frac{f_x g_x}{f_x g_y - f_y g_x}\right]_{\text{at } x = x_n}.$$

else set $m = m+1$, $p = p+1$ and go to Step 2.

Step 4: If $|x_{n+1} - x_n| < \varepsilon$ then stop.

Step 5: Set $n = n+1$ and go to step 2.

2.3 Convergence of the INR Method

Without any loss of generality, setting $x_n = \xi + \varepsilon_n$ and $y_n = \eta + \varepsilon_n$, where ε_n is the error at the nth step, we have by Taylor's series

$$f(x_n, y_n) + (\xi - x_n) f_x(x_n, y_n) + (\eta - y_n) f_y(x_n, y_n) + \frac{1}{2!}\left((\xi - x_n)\frac{\partial}{\partial x} + (\eta - y_n)\frac{\partial}{\partial y}\right)^2 f(x_n, y_n)....$$

$$= g(x_n, y_n) + (\xi - x_n) g_x(x_n, y_n) + (\eta - y_n) g_y(x_n, y_n) + \frac{1}{2!}\left((\xi - x_n)\frac{\partial}{\partial x} + (\eta - y_n)\frac{\partial}{\partial y}\right)^2 g(x_n, y_n)....$$

Or,

$$y_n^{(f)} - y_n^{(g)} + (\xi - x_n)(f_x - g_x) + (\eta - y_n)(f_x - g_x) + \frac{1}{2!}\left((\xi - x_n)\frac{\partial}{\partial x} + (\eta - y_n)\frac{\partial}{\partial y}\right)^2 (f - g) + ... = 0$$

Or, $-\dfrac{y_n^{(f)} - y_n^{(g)}}{f_x g_y - f_y g_x} f_x g_x = -\varepsilon_n (f_x + g_x) + \dfrac{f_x^2 (g_x + g_y) - (f_x + f_y) g_x^2}{f_x g_y - f_y g_x}\varepsilon_n + ...$

Or, $x_{n+1} - x_n = -\varepsilon_n (f_x + g_x) + \dfrac{f_x^2 (g_x + g_y) - (f_x + f_y) g_x^2}{f_x g_y - f_y g_x}\varepsilon_n + ...$

Or, $\varepsilon_{n+1} \approx \varepsilon_n \left[(1 - f_x - g_x) + \dfrac{f_x^2 (g_x + g_y) - (f_x + f_y) g_x^2}{f_x g_y - f_y g_x}\right]$

Therefore,

$$\frac{\varepsilon_{n+1}}{\varepsilon_n} \rightarrow \text{ a finite value,}$$

Provided

$$f_x g_y - f_y g_x \neq 0.$$

Hence, the iteration converges invariably in keeping with expectation.
Note: To get better convergence rate, consider 'x' (mentioned in the above illustration) as the variable which occurs in nonlinear form in both the equations.

3. NUMERICAL EXAMPLES FOR THE TEST OF CONVERGENCE RATE OF THE IMPROVED METHOD

In order to assess the convergence of the proposed version of improved Newton-Raphson method we have carried out a number of convergence tests on computer. We have used a number of real simultaneous analytic functions in testing of the method, including algebraic (or polynomial) and transcendental functions. In all of our examples, the maximum number of iteration is $n = 200$ and we tested these examples with precision $\varepsilon = 1\times 10^{-5}$.

The following table represents a comparison study of iteration number between the improved Newton Method by Hardaway [3] and our proposed improved Newton-Raphson method.

Simultaneous Nonlinear Equation	Initial Approximations	Number of iteration		(x_n, y_n)
		Hardaway [3]	Present analysis	
$f = 10(y - x^2)$ $g = 1 - x - y$	$(0,0)$	21	3	$(0.61803, 0.38196)$
$f = x^2 + y^2 - 1.0$ $g = 0.75x^3 - y + 0.9$	$(-0.4, -0.1)$	15	6	$(-0.98170, -0.19042)$
$f = x^2 + y^2 - 1.0$ $g = 0.75x^3 - y + 0.9$	$(1.3, -0.3)$	11	4	$(0.35697, 0.93412)$
$f = 2x + y - 7$ $g = \cos 2x - \ln(1+y) + 2$	$(1,1)$	7	5	$(0.86844, 5.26312)$
$f = x - y$ $g = \exp(-2x^2 + y) - 0.5$	$(1,1)$	8	6	$(0.69315, 0.69315)$

Table 1: The number iteration of Hardaway [3] and present analysis for a specified order of precision.

4. RESULTS AND CONCLUSION

The algorithm presented here can be programmed for a digital computer with relative ease. The speed of computation, the number of iterations and the accuracy of the solution compare very favorably with other modified Newton's methods in current use. The new approach offers some advantages:

1. In this method we do not have to calculate the Jacobian matrix and its inversion at each iteration.
2. We need not have to assume non-singularity of the Jacobian at the initial approximation.

Finally, this algorithm may be extended for three or more variables.

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REFERENCES

[1] BEN-ISRAEL, A. (1965): A modified Newton Raphson method for the solution of system of equations. **Israel J. Math**. 3, 94-98.

[2] BROYDEN, C.G. (1965): A class of methods for solving nonlinear simultaneous equations. **Math. Comput**. 19, 577-593.

[3] HARDAWAY, R.H. (1968): An algorithm for finding a solution of simultaneous nonlinear equations. **Fall Joint Computer Conference.** pp. 105-113.

[4] HENRICI, P. (1964): **Elements of Numerical Analysis**. John Wiley & Sons, Inc. N. York.

[5] HART. W.L. and MOTZKIN, T.S. (1956): A composite Newton-Raphson gradient method for the solution of system of equations. **Pacific J. Math**. 6, 691-707.

[6] IBIDAPO-OBE, O., ASAOLU, O.S. and BADIRU, A.B. (2002): A new method for the numerical solution of simultaneous nonlinear equation. **Appl. Math. Compt**. 125, 133-140.
[7] OSTROWSKI, A.M. (1960): **Solution of Equations and System of Equations**. Academic Press, N. York.
[8] PRESS, W.H., TEUKOLSKY, S.A., VETTERLING, W.T. and FLANNERY, B.P. (1992): **Numerical Recipes.** Cambridge University Press, Cambridge, N. York.
[9] SCHEFFEL, J. and HÅKANSSON, C. (2009): Solution of systems of nonlinear equations – a semi-implicit approach. **Appl. Num. Math**. 59, 1430-2443.
[10] SHAW, S. and MUKHOPADHYAY, B. (2015): An improved Regula falsi method for finding simple roots of nonlinear equations. **Appl. Math. Compt**. 254, 370-374.
[11] STOER, J. and BULIRSCH, R. (1993): **Introduction to Numerical Analysis**. Springer, N.York.