TWO-WAREHOUSE INVENTORY MODEL FOR NON-INSTANTANEOUS DETERIORATING ITEMS UNDER DIFFERENT DISPATCH POLICIES

Chandra K. Jaggi[*1], Satish K. Goel, Sunil Tiwari
Department of Operational Research
Faculty of Mathematical Sciences
New Academic Block
University of Delhi, Delhi 110007, India

ABSTRACT
A two-warehouse inventory problem for non-instantaneous deteriorating items with constant demand rate under different dispatching policies is premeditated. While formulating the inventory model for deteriorating items, usually it is assumed that the items start deteriorating as soon as they enter into the warehouse. However, there are numerous products like dry fruits, food grains etc. that have a shelf-life and start deteriorating after a time lag that is termed as non-instantaneous deterioration. Moreover, at times there are situations like eye catching price discount, low cost storage, huge demand etc. and under such a situation one may decide to procure large quantity of the items which would arise the problem of storing. As the capacity of own warehouse is limited, therefore one has to hire another warehouse to store the excess quantity. To incorporate above scenario, a two-warehouse inventory model for non-instantaneous deteriorating item under different dispatching policy i.e. LIFO, FIFO has been developed. Shortages are allowed and partially backlogged. A simple solution procedure has been provided to determine the optimal replenishment schedule. Further, the comparative study of two models i.e. LIFO and FIFO has been performed followed by the sensitivity analysis of the optimal solution with respect to major parameters is also carried out.

KEYWORDS: Inventory; non-instantaneous deterioration; two-warehouse; partial backlogging; LIFO; FIFO.

MSC: 90B13

RESUMEN
Un problema de inventario para dos almacenes de productos con deterioro no instantáneo, con tasa de demanda constante, bajo diferentes políticas de despacho es presentado. Al formular el modelo de inventario para productos con deterioro, usualmente se asume que los productos comienzan a deteriorarse tan pronto entran en el almacén. Sin embargo hay numerosos productos como las frutas secas, granos comestibles etc. que poseen una vida de mostrador y comienzan su deterioro después de un tiempo el que es llamado de deterioro no-instantáneo. Más aun, en ciertos momentos hay situaciones como el atraer la atención con precios de descuento, bajo costo de almacenaje para la gran demanda etc. y bajo tal situación se puede decidir procurar grandes cantidades de productos, los que pueden ser generados por problemas con el almacenaje. Como la capacidad del almacén propio es limitado, se tiene que alquilar otro almacén para almacenar las cantidades en exceso. Al incorporar este escenario, un modelo de inventario de dos almacenes para productos con deterioro no instantáneo bajo diferentes políticas de despacho i.e. LIFO, FIFO, han sido desarrollados. Se permite la existencia de carencias y devoluciones parciales. Un procedimiento simple de solución ha sido desarrollado para determinar el schedule de reabastecimiento optimal. Además, el estudio comparativo de dos modelos i.e. LIFO and FIFO es llevado a cabo seguido de un análisis de las soluciones óptimas respecto a la solución optimal respecto a los más importantes parámetros son llevados acabo.

1. INTRODUCTION

The problem of managing deteriorating inventory has received a considerable attention in recent years. Generally, deterioration is defined as damage, spoilage, decay, obsolescence, evaporation, pilferage, etc., that result in decreasing the usefulness of the original one. There is hardly any need for considering the effect of deterioration in the determination of the economic lot size for the items having low deterioration rate such as steel, hardware, glassware, toys, etc. However, items such as food items, pharmaceuticals, chemicals, blood, alcohol, gasoline, radioactive etc. deteriorate very rapidly over time and the loss from deterioration in these items cannot be ignored. Ghare and Schrader [1] presented an Economic Order Quantity (EOQ) model for deteriorating items assuming exponential decay. Covert and Phillip [2] extended

[1] Corresponding Author. Tel Fax: 91-11-27666672
E-mail: ckjaggi@yahoo.com, satishgoel2@hotmail.com, sunil.tiwari047@gmail.com

the Ghare and Schrader [1] model with the assumption of Weibull distribution deterioration. Three surveys on trends in modeling of continuously deteriorating inventory are those by Raafat et al [3], Goyal and Giri, [4]. Thereafter, many other authors have also done interesting work in this area of deterioration.

In all the above models, the non-instantaneous deterioration was not considered. The term "non-instantaneous deteriorating item" refers that an item retains its quality or freshness for some extent of time after which it loses its usefulness from the original condition. In other words, for non-instantaneous deteriorating item deterioration does not occur prior to certain period of time. This characteristic can be usually observed in almost all food stuffs, fashionable items, electronics products etc. At first, Wu et al. [5] introduced the phenomenon "non-instantaneous deterioration" and established the optimal replenishment policy for non-instantaneous deteriorating item with stock dependent demand and partial backlogging. Subsequently, many researchers such as Ouyang et al. [6, 7], Wu et al. [8], Jaggi and Verma [9], Chang et al. [10], Geetha et al. [11], Soni et al. [12], Maihami and Kamalabadi [13,14], Shah et al. [15], Dye [16] have studied the inventory models for non-instantaneous deteriorating items under variety of conditions.

Important aspect associated with inventory management is to decide where to store the goods, when large stock has to be procured. There are many such situations which require additional storage facility referred as Rented Warehouse (RW). An early discussion on the effect of two warehouse was considered by Hartely [17] in which he assumed that the holding cost in rented warehouse (RW) is greater than that in own warehouse (OW), therefore, items in RW are first transferred to OW to meet the demand until the stock level in RW drops to zero and then items in OW are released. Sarma [18] extended Hartley's model to cover the transportation cost from RW to OW that is considered to be a fixed constant independent of the quantity being transported. But he did not consider shortages in his model. Goswami and Chaudhuri [19] further developed the model with or without shortages by assuming that the demand varies over time with linearly increasing trend and that the transportation cost from RW to OW depends on the quantity being transported. In their model, the stock was transferred from RW to OW in an intermittent pattern. However, their work is for non-deteriorating items. In addition, a great deal of research efforts has been devoted to inventory models of deteriorating items in two warehousing area. Sarma [18] developed a two-warehouse model for deteriorating items with the infinite replenishment rate and shortages. Pakkala and Achary [20] further considered the two-warehouse model for deteriorating items with finite replenishment rate and shortages. Bhunia and Maiti [21] developed a two-warehouse model for deteriorating items with linearly increasing demand and shortages during the infinite period. Recently, Jaggi and Verma [22] have investigated the effect of deterioration with two storage facilities under FIFO dispatching policy and compared it with Sarma's, [18] LIFO model. Research continues with Zhou, [23]; Chung and Huang, [24], Das et al. [25], Dye et al. [26], Niu and Xie [27], Rong et al. [28], Lee [29], Lee and Hsu [30] and many more.

In this paper, a two warehouse inventory model for non-instantaneous deteriorating items has been developed, when shortages are partially backlogged. Further the application of FIFO and LIFO dispatching policies has been investigated. The basic objective of this work is to determine the optimal replenishment policy which minimizes the total average cost. Hence, comparison between FIFO and LIFO dispatch policies has been exhibited with the help of a numerical example. A comprehensive sensitivity analysis of the optimal solution with respect to major parameters is also carried out.

2. ASSUMPTION AND NOTATIONS

The following assumptions and notations have been used in the entire paper.

2.1. Assumptions

(i) Replenishment rate is instantaneous.
(ii) Lead-time is negligible.
(iii) The planning horizon of the inventory system is infinite.
(iv) t_d is the length of time during which the product has no deterioration.
(v) The OW has a fixed capacity of W units; the RW has unlimited capacity.
(vi) The unit inventory holding cost per unit time in RW is higher than that in OW and the deterioration rate in RW is less than that in OW.
(vii) Unsatisfied demand/shortages are allowed. Unsatisfied demand is partially backlogged and the fraction of shortages backlogged is a differentiable and decreasing function of time t, denoted by $g(t)$, where *t* is the waiting time up to the next replenishment. We have defined the partial backlogging rate $g(t) = e^{-\delta t}$, where δ is a positive constant.

2.2. Notations

In addition, the following notations are used throughout this paper.

A	replenishment cost per order
c	purchasing cost per unit
W	capacity of the owned warehouse
D	demand rate per unit time
Q_F, Q_L	order quantity per cycle for FIFO and LIFO respectively
S_F, S_L	maximum inventory level per cycle for FIFO and LIFO respectively
H	holding cost per unit per unit time in OW
F	holding cost per unit per unit time in RW, where $F > H$
s	the backlogging cost per unit per unit time, if shortage is backlogged
c_1	unit opportunity cost due to lost sale, if the shortage is lost
A	deterioration rate in OW, where $0 \le \alpha < 1$
β	deterioration rate in RW, where $0 \le \beta < 1$; $\beta < \alpha$.
t_d	time period during which no deterioration occurs
t_r	time at which the inventory level reaches zero in RW
t_w	time at which the inventory level reaches zero in OW
T	the length of the replenishment cycle in year
$I_0(t)$	inventory level in the OW at any time t where $0 \le t \le T$
$I_r(t)$	inventory level in the RW at any time t where $0 \le t \le T$
TC_i	total relevant cost per unit time for case $i=1, 2$
$B(t)$	backlogged level at any time t where $t_w \le t \le T$
$L(t)$	number of lost sales at any time t where $t_w \le t \le T$

3. MODEL DESCRIPTION AND ANALYSIS

In the present study a two warehouse inventory model has been developed, where the OW has a fixed capacity of W units and the RW has unlimited capacity. The units in RW are stored only when the capacity of OW has been utilized completely. Demand is assumed to be constant. Shortages are allowed but are partially backlogged. Moreover, in such a scenario organization has an option to adopt either LIFO or FIFO dispatching policy. In Last in first out (LIFO) approach the goods are stored in Owned Warehouse(OW) initially after satisfying the OW, remaining goods are stored in Rented Warehouse(RW) but uses the goods of RW prior to the goods of OW to satisfy the demand in order to reduce the inventory carrying charge(holding cost). Whereas in First in first out (FIFO) approach those goods are sold that are stored first in order to maintain the freshness of product which results in greater customer satisfaction. Which ultimately boost the sales and increase the value of the organisation in the long term. The following sections discuss the model formulation for both the policies.

3.1. FIFO model formulation

In this section we discuss an inventory system adopting FIFO policy. Initially a lot size of Q_F units enters the system. After meeting the backorders, S_F units enter the inventory system, out of which W units are kept in OW and the remaining (S_F - W) units are kept in the RW but uses the goods of OW prior to the goods of RW to satisfy the demand in order to maintain the freshness of product which results in greater customer satisfaction. As the deterioration of item is non-instantaneous, so initially, the units do not deteriorate for some period and after that the deterioration begins. Broadly there can be two cases. Firstly, when t_d (time during which no deterioration occurs) is less than t_w (time during which inventory in OW reaches zero) and secondly, when t_d is greater than t_w.

Case 1: When $t_d < t_w$

During the time interval [$0, t_d$], there is no deterioration. So, the inventory in OW $I_o(t)$ is depleted only due to demand whereas in RW, inventory level remains the same. Further, during the time interval [t_d, t_w] the inventory level in OW $I_o(t)$ is dropping to zero due to the combined effect of demand and deterioration and the inventory in RW $I_r(t)$ gets depleted due to deterioration only. Now, during the time interval [t_w, t_r] depletion of inventory $I_r(t)$ occurs in RW due to the combined effect of demand and deterioration and it reaches to zero at time t_r. Moreover, during the interval [t_r, T] the demand is backlogged. So, $B(t)$ represents the level of negative inventory at time t during the interval [t_r, T]. The behaviour of the model over the time interval [0, T] has been represented graphically in figure 1.

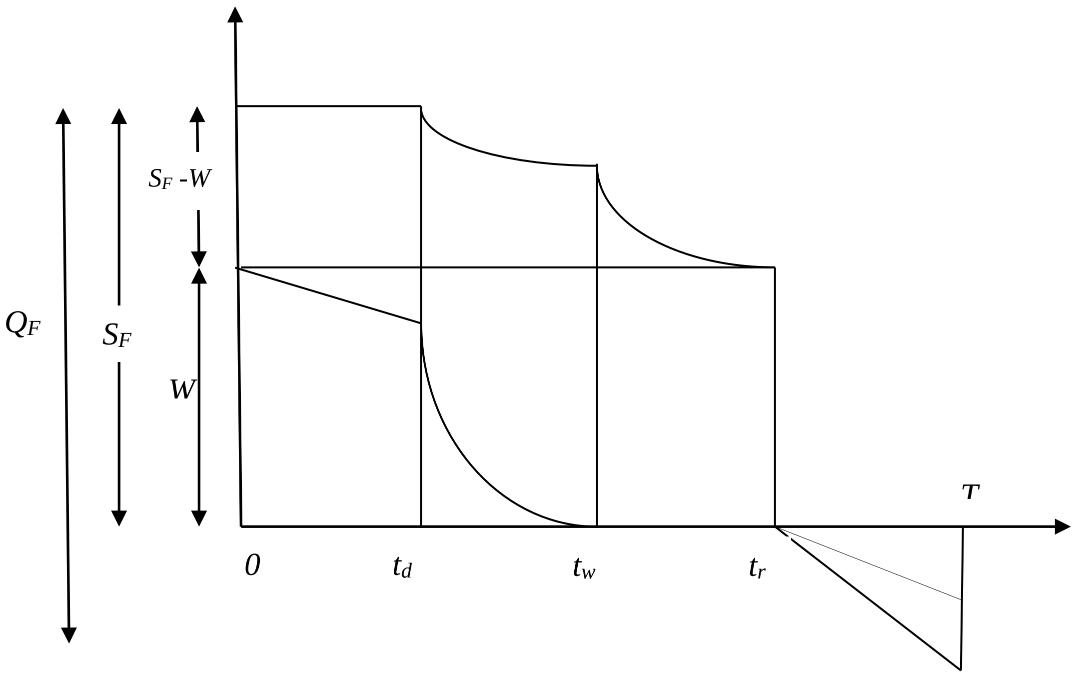


Figure 1: Two-warehouse FIFO inventory system, when $t_d < t_w$

Therefore, the differential equations that describe the inventory level in the RW and OW at time t over the period (0, T) are given by:

$$\frac{dI_0(t)}{dt} = -D \; ; \qquad for\ 0 \le t \le t_d \tag{1}$$

$$\frac{dI_0(t)}{dt} + \alpha I_0(t) = -D \; ; \qquad for\ t_d \le t \le t_w \tag{2}$$

$$\frac{dI_r(t)}{dt} + \beta I_r(t) = 0 \; ; \qquad for\ t_d \le t \le t_w \tag{3}$$

$$\frac{dI_r(t)}{dt} + \beta I_r(t) = -D \; ; \qquad for\ t_w \le t \le t_r \tag{4}$$

$$\frac{dB(t)}{dt} = De^{-\delta(T-t)} \; ; \qquad for\ t_r \le t \le T \tag{5}$$

The solutions of the above five differential equations (1), (2), (3), (4) and (5) with boundary conditions $I_0(0) = W,\ I_0(t_w) = W,\ I_r(t_d) = S_f - W,\ I_r(t_r) = 0\ and\ B(t_r) = 0$ respectively are

$$I_0(t) = W - Dt \; ; \qquad for\ 0 \le t \le t_d \tag{6}$$

$$I_0(t) = \frac{D}{\alpha}\left(e^{\alpha(t_w - t)} - 1\right) \; ; \qquad for\ t_d \le t \le t_w \tag{7}$$

$$I_r(t) = (S_f - W)e^{\beta(t_d - t)} \; ; \qquad for\ t_d \le t \le t_w \tag{8}$$

$$I_r(t) = \frac{D}{\beta}\left(e^{\beta(t_r - t)} - 1\right) \; ; \qquad for\ t_w \le t \le t_r \tag{9}$$

$$B(t)=\frac{D}{\delta}\left\{e^{-\delta(T-t)}-e^{-\delta(T-t_r)}\right\} \quad ; \qquad for\ t_r \le t \le T \tag{10}$$

The Number of lost sales at time *t* is $L(t)=\int_{t_r}^{t} D\left\{1-e^{-\delta(T-t)}\right\}dt;\ \ t_r \le t \le T$

$$=D\left[\left(t-t_r\right)-\frac{1}{\delta}\left\{e^{-\delta(T-t)}-e^{-\delta(T-t_r)}\right\}\right] \tag{11}$$

Considering continuity of $I_0(t)\ at\ t=t_d$, it follows from equations (6) and (7) that

$$t_w=t_d+\frac{1}{\alpha}\ln\left|1+\frac{\alpha}{D}\left(W-Dt_d\right)\right| \tag{12}$$

Considering continuity of $I_r(t)\ at\ t=t_w$, it follows from equations (8) and (9) that

$$\left(S_f-W\right)e^{\beta(t_d-t_w)}=\frac{D}{\beta}\left(e^{\beta(t_r-t_w)}-1\right)$$

which implies that the maximum inventory level per cycle is

$$S_f=W+\frac{D}{\beta}\left(e^{\beta(t_r-t_d)}-e^{\beta(t_w-t_d)}\right) \tag{13}$$

Putting *t* = *T* in equation (10), the maximum amount of demand backlogged per cycle is

$$B(T)=\frac{D}{\delta}\left(1-e^{-\delta(T-t_r)}\right) \tag{14}$$

Therefore, the order quantity over the replenishment cycle can be determined as

$$Q_f(t)=S_f+B(T)$$

Using equations (12) and (14)

$$Q_f(t)=W+\frac{D}{\beta}\left(e^{\beta(t_r-t_d)}-e^{\beta(t_w-t_d)}\right)+\frac{D}{\delta}\left(1-e^{-\delta(T-t_r)}\right) \tag{15}$$

Hence, the various costs during the cycle (*0*, *T*) are evaluated as follows:

a) Ordering cost per cycle = *A*

b) The inventory holding cost per cycle in RW

$$HC_{rw}=F\left[\int_0^{t_d} I_r(t)dt+\int_{t_d}^{t_w} I_r(t)dt+\int_{t_w}^{t_r} I_r(t)dt\right]$$

$$=\frac{FD}{\beta}\left[\left(e^{\beta(t_r-t_d)}-e^{\beta(t_w-t_d)}\right)\left\{t_d+\frac{1}{\beta}\left(1-e^{\beta(t_d-t_w)}\right)\right\}+\left\{\frac{1}{\beta}\left(e^{\beta(t_r-t_w)}-1\right)-\left(t_r-t_w\right)\right\}\right]$$

c) The inventory holding cost per cycle in OW

$$HC_{ow}=H\left[\int_0^{t_d} I_0(t)dt+\int_{t_d}^{t_w} I_0(t)dt\right]$$

$$HC_{ow}=H\left[Wt_d-\frac{Dt_d^2}{2}-\frac{D}{\alpha}\left\{\frac{1}{\alpha}\left(1-e^{\alpha(t_w-t_d)}\right)+\left(t_w-t_d\right)\right\}\right]$$

d) The backlogged cost per cycle is $=s\int_{t_r}^{T} B(t)\,dt$

$$SC=s\frac{D}{\delta}\left[\frac{1}{\delta}-\left\{\frac{1}{\delta}+T-t_r\right\}e^{-\delta(T-t_r)}\right]$$

e) The opportunity cost due to lost sale is $=c_1\int_{t_r}^{T}\left\{1-e^{-\delta(T-t)}\right\}Ddt$

$$= c_1 D\left[T - t_r - \frac{1}{\delta}\left\{1 - e^{-\delta(T-t_r)}\right\}\right]$$

f) The deterioration cost per cycle $= c\left[\beta\int_{t_d}^{t_r} I_r(t)dt + \alpha\int_{t_d}^{t_w} I_0(t)dt\right]$

$$= cD\left[\frac{1}{\alpha}\left(1 - e^{\alpha(t_w - t_d)}\right) + \left(t_w - t_d\right) + \left\{\frac{1}{\beta}\left(e^{\beta(t_r - t_d)} - e^{\beta(t_w - t_d)}\right)\left(1 - e^{\beta(t_d - t_w)}\right)\right.\right.$$
$$\left.\left. + \left(\frac{1}{\beta}\left(e^{\beta(t_r - t_w)} - 1\right) - \left(t_r - t_w\right)\right)\right\}\right]$$

Now, the total relevant cost per unit time during the cycle (0, *T*) using equations is given by

$$TC_{f1}(t_r, T) = \frac{1}{T}\left[A + \frac{FD}{\beta}\left\{\left(e^{\beta(t_r - t_d)} - e^{\beta(t_w - t_d)}\right)\left\{t_d + \frac{1}{\beta}\left(1 - e^{\beta(t_d - t_w)}\right)\right\} + \left\{\frac{1}{\beta}\left(e^{\beta(t_r - t_w)} - 1\right) - \left(t_r - t_w\right)\right\}\right\}\right.$$
$$+ H\left\{Wt_d - \frac{Dt_d^2}{2} - \frac{D}{\alpha}\left\{\frac{1}{\alpha}\left(1 - e^{\alpha(t_w - t_d)}\right) + \left(t_w - t_d\right)\right\}\right\} + s\frac{D}{\delta}\left\{\frac{1}{\delta} - \left(\frac{1}{\delta} + T - t_r\right)e^{-\delta(T - t_r)}\right\}$$
$$+ c_1 D\left\{T - t_r - \frac{1}{\delta}\left\{1 - e^{-\delta(T - t_r)}\right\}\right\} + cD\left\{\frac{1}{\alpha}\left(1 - e^{\alpha(t_w - t_d)}\right) + \left(t_w - t_d\right)\right.$$
$$\left.\left. + \frac{1}{\beta}\left(e^{\beta(t_r - t_d)} - e^{\beta(t_w - t_d)}\right)\left(1 - e^{\beta(t_d - t_w)}\right) + \left(\frac{1}{\beta}\left(e^{\beta(t_r - t_w)} - 1\right) - \left(t_r - t_w\right)\right)\right\}\right] \tag{16}$$

Case 2: When $t_d > t_w$

In this case, time during which no deterioration occurs is greater than the time during which inventory in OW becomes zero and the behaviour of the model over the time interval [0, *T*] has been graphically represented below in Figure 2.

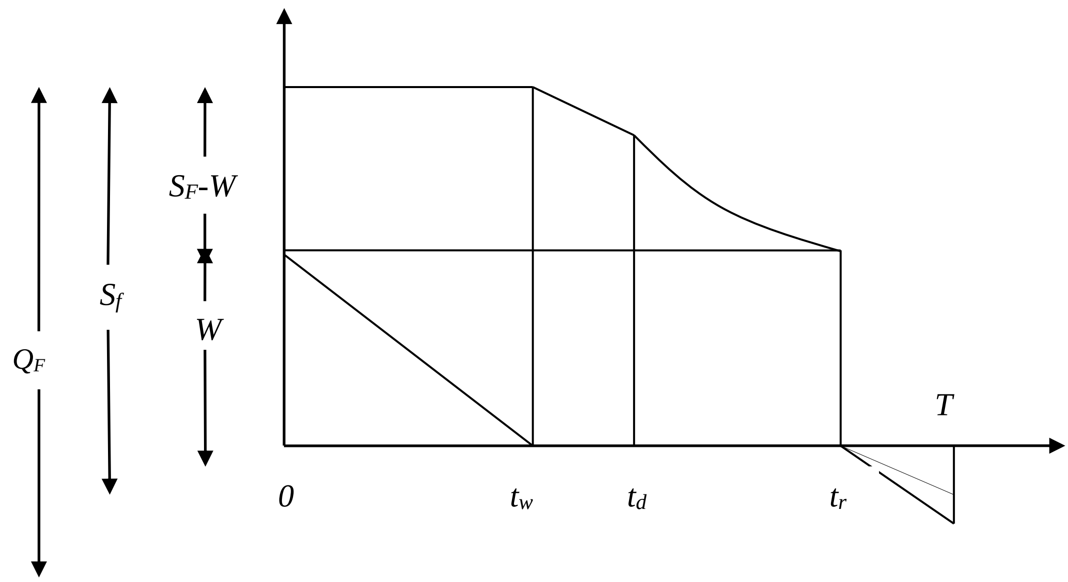


Figure 2: Two-warehouse FIFO inventory system when $t_d > t_w$

Therefore, the differential equations that describe the inventory level in the RW and OW at time *t* over the period (0, *T*) are given by:

$$\frac{dI_0(t)}{dt} = -D \ ; \qquad for\ 0 \le t \le t_w \tag{17}$$

$$\frac{dI_r(t)}{dt} = -D \ ; \qquad for\ t_w \le t \le t_d \tag{18}$$

$$\frac{dI_r(t)}{dt} + \beta I_r(t) = -D \ ; \qquad for\ t_d \le t \le t_r \tag{19}$$

$$\frac{dB(t)}{dt} = De^{-\delta(T-t)} \ ; \qquad for\ t_r \le t \le T \tag{20}$$

The solutions of the above five differential equations (17), (18), (19), and (20) with boundary conditions $I_0(0) = W,\ I_r(t_w) = S_f - W,\ I_r(t_r) = 0\ \ and\ B(t_r) = 0$ respectively are

$$I_0(t) = W - Dt \ ; \qquad for\ 0 \le t \le t_w \tag{21}$$

$$I_r(t) = D(t_w - t) + (S_f - W) \ ; \qquad for\ t_w \le t \le t_d \tag{22}$$

$$I_r(t) = \frac{D}{\beta}\left(e^{\beta(t_r - t)} - 1\right) \ ; \qquad for\ t_d \le t \le t_r \tag{23}$$

$$B(t) = \frac{D}{\delta}\left\{e^{-\delta(T-t)} - e^{-\delta(T-t_r)}\right\} \ ; \qquad for\ t_r \le t \le T \tag{24}$$

The Number of lost sales at time *t* is $L(t) = \int_{t_r}^{t} D\left\{1 - e^{-\delta(T-t)}\right\} dt;\ \ t_r \le t \le T$

$$= D\left[(t - t_r) - \frac{1}{\delta}\left\{e^{-\delta(T-t)} - e^{-\delta(T-t_r)}\right\}\right] \tag{25}$$

Now, at $t = t_w$ when $I_0(t) = 0$ we get $t_w = \dfrac{W}{D}$

Considering continuity of $I_r(t)$ at $t = t_d$, it follows from equations (22) and (23) that

$$D(t_w - t_d) + (S_f - W) = \frac{D}{\beta}\left(e^{\beta(t_r - t_d)} - 1\right)$$

which implies that the maximum inventory level per cycle is

$$S_f = W + \frac{D}{\beta}\left(e^{\beta(t_r - t_d)} - 1\right) + D(t_d - t_w) \tag{26}$$

Putting $t = T$ in equation (24), the maximum amount of demand backlogged per cycle is

$$B(T) = \frac{D}{\delta}\left(1 - e^{-\delta(T-t_r)}\right) \tag{27}$$

Therefore, the order quantity over the replenishment cycle can be determined as

$$Q_f = S_f + B(T)$$

Using equations (26) and (27)

$$Q_f = W + \frac{D}{\beta}\left(e^{\beta(t_r - t_d)} - 1\right) + D(t_d - t_w) + \frac{D}{\delta}\left(1 - e^{-\delta(T-t_r)}\right) \tag{28}$$

The total cost per cycle consists of the following elements:

a) Ordering cost per cycle = *A*
b) The inventory holding cost per cycle in RW

$$HC_{rw} = F\left[\int_0^{t_w} I_r(t)dt + \int_{t_w}^{t_d} I_r(t)dt + \int_{t_d}^{t_r} I_r(t)dt\right]$$

$$HC_{rw}=FD\left[\left\{\frac{1}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)+\left(t_d-t_w\right)\right\}t_d+\left\{\frac{1}{\beta^2}\left(e^{\beta(t_r-t_d)}-1\right)-\frac{\left(t_d-t_w\right)^2}{2}-\frac{\left(t_r-t_d\right)}{\beta}\right\}\right]$$

c) The inventory holding cost per cycle in OW

$$HC_{ow} = H\int_0^{t_w} I_0(t)dt = \frac{HDt_w^2}{2}$$

d) The backlogged cost per cycle is $= s\int_{t_r}^{T} B(t)\,dt$

$$SC = s\frac{D}{\delta}\left[\frac{1}{\delta}-\left\{\frac{1}{\delta}+T-t_r\right\}e^{-\delta(T-t_r)}\right]$$

e) The opportunity cost due to lost sale is $= c_1\int_{t_r}^{T}\left\{1-e^{-\delta(T-t)}\right\}Ddt$

$$= c_1D\left[T-t_r-\frac{1}{\delta}\left\{1-e^{-\delta(T-t_r)}\right\}\right]$$

f) The deterioration cost per cycle $= c\beta\int_{t_d}^{t_r} I_r(t)dt$

$$= cD\left\{\frac{1}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)-\left(t_r-t_d\right)\right\}$$

Now, the total relevant cost per unit time during the cycle (0, *T*) using equations is given by

$$TC_{f2}(t_r,T)=\frac{1}{T}\left[A+\frac{HDt_w^2}{2}+FD\left\{\left\{\frac{1}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)+\left(t_d-t_w\right)\right\}t_d+\left\{\frac{1}{\beta^2}\left(e^{\beta(t_r-t_d)}-1\right)-\frac{\left(t_d-t_w\right)^2}{2}-\frac{\left(t_r-t_d\right)}{\beta}\right\}\right\}\right.$$
$$\left.+\frac{sD}{\delta}\left\{\frac{1}{\delta}-\left(\frac{1}{\delta}+T-t_r\right)e^{-\delta(T-t_r)}\right\}+c_1D\left\{T-t_r-\frac{1}{\delta}\left(1-e^{-\delta(T-t_r)}\right)\right\}+cD\left\{\frac{1}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)-\left(t_r-t_d\right)\right\}\right] \quad (29)$$

Therefore, the total relevant cost per unit time during the cycle (0, *T*) is given by

$$TC_f(t_r,T)=\begin{cases}TC_{f1}(t_r,T) & if \quad t_d\le t_w\\ TC_{f2}(t_r,T) & if \quad t_d\ge t_w\end{cases} \quad (30)$$

which is a function of two continuous variable t_r and T.

Optimality

Our problem is to determine the optimum value of t_r and T which minimizes $TC_f\,(t_r, T)$. The necessary conditions for minimization of the total cost function given by equations (30) are $\frac{\partial TC_{fi}(t_r,T)}{\partial t_r}=0,\ and\ \frac{\partial TC_{fi}(t_r,T)}{\partial T}=0$ for $i = 1, 2$ which gives

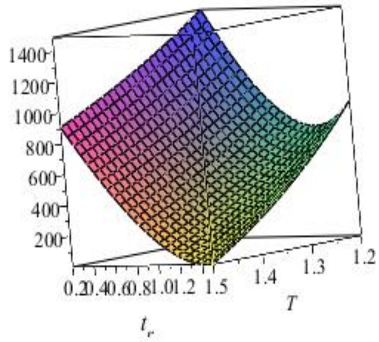
$$\frac{\partial TC_{f1}(t_r,T)}{\partial t_r}=\frac{D}{T}\left[\left(Ft_d+c\beta\right)e^{\beta(t_r-t_d)}+\frac{F}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)+\left\{c_1-s\left(T-t_r\right)\right\}e^{-\delta(T-t_r)}-c_1\right]=0 \quad (31.a)$$

$$
\begin{aligned}
\frac{\partial TC_{f1}(t_r, T)}{\partial T} = & -\frac{1}{T^2}\Bigg[\frac{FD}{\beta}\Bigg\{\left(e^{\beta(t_r-t_d)}-e^{\beta(t_w-t_d)}\right)\left\{t_d+\frac{1}{\beta}\left(1-e^{\beta(t_d-t_w)}\right)\right\}+\left\{\frac{1}{\beta}\left(e^{\beta(t_r-t_w)}-1\right)-\left(t_r-t_w\right)\right\}\Bigg\} \\
& +H\left\{Wt_d-\frac{Dt_d^2}{2}-\frac{D}{\alpha}\left\{\frac{1}{\alpha}\left(1-e^{\alpha(t_w-t_d)}\right)+\left(t_w-t_d\right)\right\}\right\}+\frac{sD}{\delta}\left\{\frac{1}{\delta}-\left(\frac{1}{\delta}+T-t_r\right)e^{-\delta(T-t_r)}\right\} \\
& +c_1D\left\{T-t_r-\frac{1}{\delta}\left\{1-e^{-\delta(T-t_r)}\right\}\right\}+cD\Bigg\{\frac{1}{\alpha}\left(1-e^{\alpha(t_w-t_d)}\right)+\left(t_w-t_d\right)+\frac{1}{\beta}\left(e^{\beta(t_r-t_d)}-e^{\beta(t_w-t_d)}\right)\left(1-e^{\beta(t_d-t_w)}\right) \\
& +\left(\frac{1}{\beta}\left(e^{\beta(t_r-t_w)}-1\right)-\left(t_r-t_w\right)\right)\Bigg\}\Bigg]+\frac{D}{T}\left\{s\left(T-t_r\right)e^{-\delta(T-t_r)}+c_1\left(1-e^{-\delta(T-t_r)}\right)\right\}=0
\end{aligned} \tag{31.b}
$$

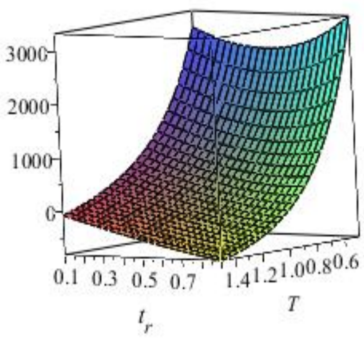
$$
\frac{\partial TC_{f2}(t_r,T)}{\partial t_r}=\frac{D}{T}\left[F\left\{e^{\beta(t_r-t_d)}t_d+\frac{1}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)\right\}-\left\{c_1+s\left(T-t_r\right)\right\}e^{-\delta(T-t_r)}+c_1+c\left(e^{\beta(t_r-t_d)}-1\right)\right]=0 \tag{31.c}
$$

and

$$
\begin{aligned}
\frac{\partial TC_2(t_r, T)}{\partial T} = & -\frac{1}{T^2}\Bigg[A+\frac{HDt_w^2}{2}+FD\left\{\left\{\frac{1}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)+\left(t_d-t_w\right)\right\}t_d+\left\{\frac{1}{\beta^2}\left(e^{\beta(t_r-t_d)}-1\right)-\frac{\left(t_d-t_w\right)^2}{2}-\frac{\left(t_r-t_d\right)}{\beta}\right\}\right\} \\
& +\frac{sD}{\delta}\left\{\frac{1}{\delta}-\left(\frac{1}{\delta}+T-t_r\right)e^{-\delta(T-t_r)}\right\}+c_1D\left\{T-t_r-\frac{1}{\delta}\left(1-e^{-\delta(T-t_r)}\right)\right\}+cD\left\{\frac{1}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)-\left(t_r-t_d\right)\right\}\Bigg] \\
& +\frac{D}{T}\left[s\left(T-t_r\right)e^{-\delta(T-t_r)}+c_1\left(1-e^{-\delta(T-t_r)}\right)\right]=0
\end{aligned} \tag{31.d}
$$



Case 1: When $t_d < t_w$



Case2: When $t_d > t_w$

Figure 3: Convexity of cost function for FIFO w.r.t. t_r and T

Equations [(31.a) and (31.b)] and [(31.c) and (31.d)] can be solved simultaneously for the optimal values of t_{ri} and T_i (say t_{ri}^* and T_i^*) for i = 1, 2 provided, it also satisfies the following sufficient conditions

$$\frac{\partial^2 TC_{fi}(t_r,T)}{\partial t_r^2}>0,\frac{\partial^2 TC_{fi}(t_r,T)}{\partial T^2}>0 \text{ and } \left[\left(\frac{\partial^2 TC_{fi}}{\partial t_r^{\,2}}\times\frac{\partial^2 TC_{fi}}{\partial T^2}\right)-\left(\frac{\partial^2 TC_{fi}}{\partial T\partial t_r}\times\frac{\partial^2 TC_{fi}}{\partial t_r\partial T}\right)\right]>0 \text{ for } i=1,2$$

Details are provided in Appendix see Appendix1.
Mathematically, it is very difficult to prove the sufficient conditions, so convexities of cost function for both cases are shown graphically in figure 3.

3.2. LIFO model formulation

In this section we discuss an inventory system adopting LIFO policy. In Last in first out (LIFO) approach the goods are stored in Owned Warehouse(OW) initially after satisfying the OW, remaining goods are stored in Rented Warehouse(RW) but uses the goods of RW prior to the goods of OW to satisfy the demand in order to reduce the to reduce the inventory carrying charge(holding cost). Initially a lot size of Q_L units enters the system. After meeting the backorders, S_L units enter the inventory system, out of which W units are kept in OW and the remaining (S_L -W) units are kept in the RW. As the deterioration of item is non-instantaneous, so initially, the units do not deteriorate for some period and after that the deterioration begins. Broadly there can be two cases. Firstly, when t_d (time during which no deterioration occurs) is less than t_r (time during which inventory in RW becomes zero) and secondly when t_d (time during which no deterioration occurs) is greater than t_r (time during which the inventory in RW becomes zero).

Case 1: When $t_d < t_r$

During the time interval $[0,\ t_d]$, there is no deterioration so the inventory in RW is depleted only due to demand whereas in OW inventory level remains the same. Further, during the time interval $[t_d,\ t_r]$ the inventory level in RW is dropping to zero due to the combined effect of demand and deterioration and the inventory in OW gets depleted due to deterioration alone. Now, during the time interval $[t_r,\ t_w]$ depletion of inventory occurs in OW due to the combined effect of demand and deterioration and it reaches to zero at time t_w. Moreover, during the interval $[t_w,\ T]$ the demand is backlogged. The behaviour of the model over the time interval $[0,\ T]$ has been graphically represented below in Figure 4.

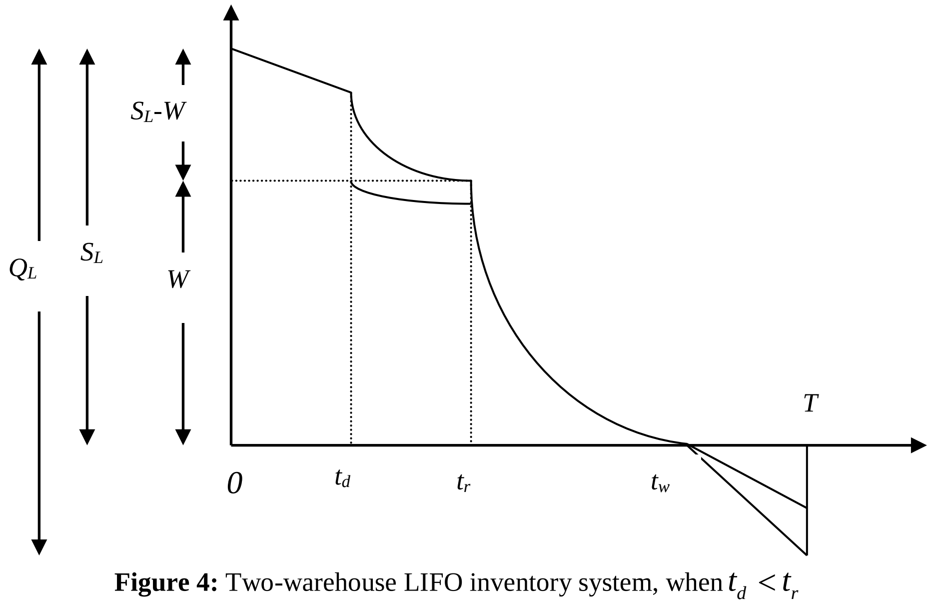


Figure 4: Two-warehouse LIFO inventory system, when $t_d < t_r$

Therefore, the differential equations that describe the inventory level in the RW and OW at time *t* over the period (*0*, *T*) are given by:

$$\frac{dI_r(t)}{dt} = -D, \qquad 0 \le t \le t_d \tag{32}$$

$$\frac{dI_r(t)}{dt} + \beta I_r(t) = -D, \qquad t_d \le t \le t_r \tag{33}$$

$$\frac{dI_0(t)}{dt} + \alpha I_0(t) = 0, \qquad t_d \le t \le t_r \tag{34}$$

$$\frac{dI_0(t)}{dt} + \alpha I_0(t) = -D, \qquad t_r \le t \le t_w \tag{35}$$

$$\frac{dB(t)}{dt} = De^{-\delta(T-t)}, \qquad t_w \le t \le T \tag{36}$$

The solutions of the above five differential equations (32), (33), (34), (35) and (36) with boundary conditions $I_r(0) = S_L - W, I_r(t_r) = 0, I_0(t_d) = W, I_0(t_w) = 0$ & $B(t_w) = 0$ respectively are

$$I_r(t) = S_L - Dt - W, \qquad 0 \le t \le t_d \tag{37}$$

$$I_r(t) = \frac{D}{\beta}\left(e^{\beta(t_r - t)} - 1\right), \qquad t_d \le t \le t_r \tag{38}$$

$$I_0(t) = We^{\alpha(t_d - t)}, \qquad t_d \le t \le t_r \tag{39}$$

$$I_0(t) = \frac{D}{\alpha}\left(e^{\alpha(t_w - t)} - 1\right), \qquad t_r \le t \le t_w \tag{40}$$

$$B(t) = \frac{D}{\delta}\left\{e^{-\delta(T-t)} - e^{-\delta(T-t_w)}\right\}, \qquad t_w \le t \le T \tag{41}$$

The Number of lost sales at time *t* is

$$L(t) = \int_{t_w}^{t} D\left\{1 - e^{-\delta(T-t)}\right\} dt; \qquad t_w \le t \le T$$

$$= D\left[(t - t_w) - \frac{1}{\delta}\left\{e^{-\delta(T-t)} - e^{-\delta(T-t_w)}\right\}\right] \tag{42}$$

Considering continuity of $I_r(t)$ at $t = t_d$, it follows from equations (37) and (38) that

$$S_L - Dt_d - W = \frac{D}{\beta}\left(e^{\beta(t_r - t_d)} - 1\right) \tag{43}$$

which implies that the maximum inventory level per cycle is

$$S_L = W + Dt_d + \frac{D}{\beta}\left(e^{\beta(t_r - t_d)} - 1\right) \tag{44}$$

Considering continuity of $I_0(t)$ at $t = t_r$ it follows from equations (39) and (40) that

$$We^{\alpha(t_d - t_r)} = \frac{D}{\alpha}\left(e^{\alpha(t_w - t_r)} - 1\right)$$

$$t_w = t_r + \frac{1}{\alpha}\ln\left(\frac{D + \alpha We^{\alpha(t_d - t_r)}}{D}\right) \tag{45}$$

Putting $t = T$ in equation (41), the maximum amount of demand backlogged per cycle is

$$B(T) = \frac{D}{\delta}\left(1 - e^{-\delta(T - t_w)}\right) \tag{46}$$

Therefore, the order quantity over the replenishment cycle can be determined as

$Q_L = S_L + B(T)$ (Using equations (44) and (46))

$$=W+Dt_d+\frac{D}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)+\frac{D}{\delta}\left(1-e^{-\delta(T-t_w)}\right) \tag{47}$$

The total cost per cycle consists of the following elements:

a) Ordering cost per cycle = A

b) The inventory holding cost per cycle in RW $=F\left(\int_0^{t_d} I_r(t)\,dt+\int_{t_d}^{t_r} I_r(t)\,dt\right)$

$$=F\left\{S_L t_d-\frac{Dt_d^2}{2}-Wt_d-\frac{D}{\beta}\left(t_r-t_d+\frac{1}{\beta}\left(1-e^{\beta(t_r-t_d)}\right)\right)\right\}$$

$$=F\left\{\frac{D}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)\left(t_d+\frac{1}{\beta}\right)+\frac{Dt_d^2}{2}-\frac{D}{\beta}(t_r-t_d)\right\}$$

c) The inventory holding cost per cycle in OW $=H\left(\int_0^{t_d} W\,dt+\int_{t_d}^{t_r} I_0(t)\,dt+\int_{t_r}^{t_w} I_0(t)\,dt\right)$

$$=H\left\{Wt_d+\frac{W}{\alpha}\left(1-e^{\alpha(t_d-t_r)}\right)+\frac{D}{\alpha}\left(t_r-t_w-\frac{1}{\alpha}\left(1-e^{\alpha(t_w-t_r)}\right)\right)\right\}$$

d) The backlogged cost per cycle is $=s\int_{t_w}^{T} B(t)\,dt$

$$SC=s\frac{D}{\delta}\left[\frac{1}{\delta}-\left\{\frac{1}{\delta}+T-t_w\right\}e^{-\delta(T-t_w)}\right]$$

e) The opportunity cost due to lost sale per cycle is $=c_1\int_{t_w}^{T}\left\{1-e^{-\delta(T-t)}\right\}D\,dt$

$$=c_1 D\left[T-t_w-\frac{1}{\delta}\left\{1-e^{-\delta(T-t_w)}\right\}\right]$$

f) The deterioration cost per cycle $=c\left[\beta\int_{t_d}^{t_r} I_r(t)\,dt+\alpha\int_{t_d}^{t_w} I_0(t)\,dt\right]$

$$=c\left\{W\left(1-e^{\alpha(t_d-t_r)}\right)+D(t_d-t_w)-D\left(\frac{1}{\alpha}\left(1-e^{\alpha(t_w-t_r)}\right)+\frac{1}{\beta}\left(1-e^{\beta(t_r-t_d)}\right)\right)\right\}$$

Now, the total relevant cost per unit time during the cycle (0, T) is given by

$$\begin{aligned}TC_{L1}(t_r,T)=\frac{1}{T}\Bigg[&A+F\left\{\frac{D}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)\left(t_d+\frac{1}{\beta}\right)+\frac{Dt_d^2}{2}-\frac{D}{\beta}(t_r-t_d)\right\}+H\left\{Wt_d+\frac{W}{\alpha}\left(1-e^{\alpha(t_d-t_r)}\right)\right.\\&\left.+\frac{D}{\alpha}\left(t_r-t_w-\frac{1}{\alpha}\left(1-e^{\alpha(t_w-t_r)}\right)\right)\right\}+s\frac{D}{\delta}\left(\frac{1}{\delta}-\left(\frac{1}{\delta}+T-t_w\right)e^{-\delta(T-t_w)}\right)+c_1D\left\{T-t_w-\frac{1}{\delta}\left(1-e^{-\delta(T-t_w)}\right)\right\}\\&+c\left\{W\left(1-e^{\alpha(t_d-t_r)}\right)+D(t_d-t_w)-D\left(\frac{1}{\alpha}\left(1-e^{\alpha(t_w-t_r)}\right)+\frac{1}{\beta}\left(1-e^{\beta(t_r-t_d)}\right)\right)\right\}\Bigg]\end{aligned} \tag{48}$$

Where $t_w=t_r+\frac{1}{\alpha}\ln\left(\frac{D+\alpha We^{\alpha(t_d-t_r)}}{D}\right)$

Case 2: When $t_d>t_r$

In this case, time during which no deterioration occurs is greater than the time during which inventory in RW becomes zero and the behaviour of the model over the whole cycle $[0,\ T]$ has been graphically represented as in Figure 5.

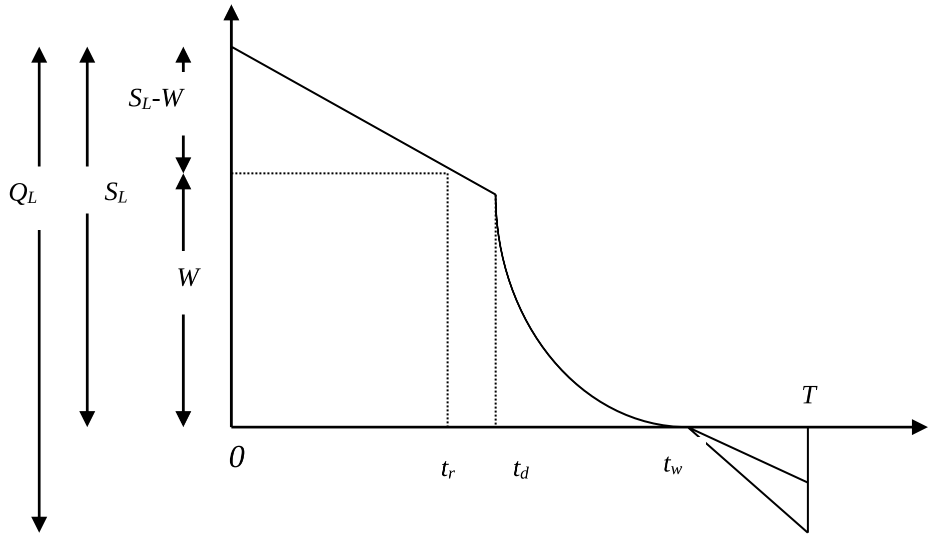


Figure 5: Two-warehouse LIFO inventory system, when $t_d > t_r$

Therefore, the differential equations that describe the inventory level in the RW and OW at time t over the period (*0, T*) are given by:

$$\frac{dI_r(t)}{dt} = -D, \qquad 0 \le t \le t_r \tag{49}$$

$$\frac{dI_0(t)}{dt} = -D, \qquad t_r \le t \le t_d \tag{50}$$

$$\frac{dI_0(t)}{dt} + \alpha I_0(t) = -D, \qquad t_d \le t \le t_w \tag{51}$$

$$\frac{dB(t)}{dt} = De^{-\delta(T-t)}, \qquad t_w \le t \le T \tag{52}$$

The solution of the above four differential equations (49), (50), (51) and (52) with boundary conditions $I_r(t_r) = 0, I_0(t_r) = W, I_0(t_W) = 0, B(t_W) = 0$ respectively are

$$I_r(t) = D(t_r - t), \qquad 0 \le t \le t_r \tag{53}$$

$$I_0(t) = W + Dt_r - Dt, \qquad t_r \le t \le t_d \tag{54}$$

$$I_0(t) = \frac{D}{\alpha}\left(e^{\alpha(t_w - t)} - 1\right), \qquad t_d \le t \le t_w \tag{55}$$

$$B(t) = \frac{D}{\delta}\left\{e^{-\delta(T-t)} - e^{-\delta(T-t_w)}\right\}, \quad t_w \le t \le T \tag{56}$$

The Number of lost sales at time *t* is

$$L(t) = \int_{t_w}^{t} D\left\{1 - e^{-\delta(T-t)}\right\} dt; \qquad t_w \le t \le T$$

$$= D\left[(t - t_w) - \frac{1}{\delta}\left\{e^{-\delta(T-t)} - e^{-\delta(T-t_w)}\right\}\right] \tag{57}$$

Considering continuity of $I_0(t)$ at $t = t_d$, it follows from equations (54) and (55) that

$$W + Dt_r - Dt_d = \frac{D}{\alpha}\left(e^{\alpha(t_w - t_d)} - 1\right) \tag{58}$$

$$t_w = t_d + \frac{1}{\alpha}\ln\left|1+\frac{\alpha W}{D}+\alpha\left(t_r - t_d\right)\right| \quad (59)$$

Now, at $t = 0$ when $I_r(t) = S_L - W$ and solving equations (53) we get the maximum inventory level is

$$S_L = W + Dt_r \quad (60)$$

Putting $t = T$ in equation (56), the maximum amount of demand backlogged per cycle is

$$B(T) = \frac{D}{\delta}\left(1-e^{-\delta(T-t_w)}\right) \quad (61)$$

Therefore, order quantity is $Q_L = S_L + B(T)$

$$= W + Dt_r + \frac{D}{\delta}\left(1-e^{-\delta(T-t_w)}\right) \quad \text{(Using equations (60) and (61))} \quad (62)$$

The total cost per cycle consists of the following elements:

a) Ordering cost per cycle = A

b) The inventory holding cost per cycle in RW $= F\left(\int_0^{t_r} I_r(t)\,dt\right) = \frac{FDt_r^2}{2}$

c) The inventory holding cost per cycle in OW $= H\left(\int_0^{t_r} W dt + \int_{t_r}^{t_d} I_0(t)\,dt + \int_{t_d}^{t_w} I_0(t)\,dt\right)$

$$= H\left[Wt_r + Dt_r\left(t_d - t_r\right) - \frac{D}{2}\left(t_d^2 - t_r^2\right) + \frac{D}{\alpha}\left\{\frac{1}{\alpha}\left(e^{\alpha(t_w - t_d)} - 1\right) - \left(t_w - t_d\right)\right\}\right]$$

d) The backlogged cost per cycle is $= s\int_{t_w}^{T} B(t)\,dt$

$$SC = s\frac{D}{\delta}\left[\frac{1}{\delta} - \left\{\frac{1}{\delta} + T - t_w\right\}e^{-\delta(T-t_w)}\right]$$

e) The opportunity cost per cycle due to lost sale is $= c_1\int_{t_w}^{T}\left\{1-e^{-\delta(T-t)}\right\}D dt$

$$= c_1 D\left[T - t_w - \frac{1}{\delta}\left\{1-e^{-\delta(T-t_w)}\right\}\right]$$

f) The deterioration cost per cycle $= c\alpha\int_{t_d}^{t_w} I_0(t)dt$

$$= cD\left\{\frac{1}{\alpha}\left(e^{\alpha(t_w - t_d)} - 1\right) - \left(t_w - t_d\right)\right\}$$

Now, the total relevant cost per unit time during the cycle (0, T) is given by

$$TC_{L2}\left(t_r, T\right) = \frac{1}{T}\left[A + \frac{FDt_r^2}{2} + H\left\{Wt_r + Dt_r\left(t_d - t_r\right) - \frac{D}{2}\left(t_d^2 - t_r^2\right) + \frac{D}{\alpha}\left\{\frac{1}{\alpha}\left(e^{\alpha(t_w - t_d)} - 1\right) - \left(t_w - t_d\right)\right\}\right\}\right.$$
$$\left. + \frac{sD}{\delta}\left\{\frac{1}{\delta} - \left(\frac{1}{\delta} + T - t_w\right)e^{-\delta(T-t_w)}\right\} + c_1 D\left\{T - t_w - \frac{1}{\delta}\left(1-e^{-\delta(T-t_w)}\right)\right\} + cD\left\{\frac{1}{\alpha}\left(e^{\alpha(t_w - t_d)} - 1\right) - \left(t_w - t_d\right)\right\}\right] \quad (63)$$

Where $t_w = t_d + \frac{1}{\alpha}\ln\left|1+\frac{\alpha W}{D}+\alpha\left(t_r - t_d\right)\right|$

Therefore, the total relevant cost per unit time during the cycle (0, T) is given by

$$TC_L(t_r,T)=\begin{cases}TC_{L1}(t_r,T) & if \quad t_d\le t_r\\ TC_{L2}(t_r,T) & if \quad t_d\ge t_r\end{cases} \tag{64}$$

which is a function of two continuous variable t_r and T.

Optimality:

Our problem is to determine the optimum value of t_r and T which minimizes $TC_L(t_r, T)$. The necessary conditions for minimization of the total cost function given by equation (64) are

$$\frac{\partial TC_{Li}(t_r,T)}{\partial t_r}=0,\ and\quad \frac{\partial TC_{Li}(t_r,T)}{\partial T}=0$$ for $i = 1, 2$ which gives

$$\frac{\partial TC_{L1}(t_r,T)}{\partial t_r}=\frac{1}{T}\left[FDe^{\beta(t_r-t_d)}\left(t_d+\frac{1}{\beta}\right)+H\left\{We^{\alpha(t_d-t_r)}+\frac{D}{\alpha}\left(1-e^{\alpha(t_w-t_r)}\right)(1-X_1)\right\}+D\left\{\left(c_1-\frac{2s}{\delta}-s(T-t_w)\right)e^{-\delta(T-t_w)}-c_1\right\}X_1\right.$$
$$\left.+c\left\{W\alpha e^{\alpha(t_d-t_r)}-DX_1-D\left(e^{\alpha(t_w-t_r)}(X_1-1)-e^{\beta(t_r-t_d)}\right)\right\}\right]=0 \tag{65.a}$$

$$\frac{\partial TC_{L1}(t_r,T)}{\partial T}=-\frac{1}{T^2}\left[A+F\left\{\frac{D}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)\left(t_d+\frac{1}{\beta}\right)+\frac{Dt_d^2}{2}-\frac{D}{\beta}(t_r-t_d)\right\}+H\left\{Wt_d+\frac{W}{\alpha}\left(1-e^{\alpha(t_d-t_r)}\right)\right.\right.$$
$$\left.+\frac{D}{\alpha}\left(t_r-t_w-\frac{1}{\alpha}\left(1-e^{\alpha(t_w-t_r)}\right)\right)\right\}+s\frac{D}{\delta}\left(\frac{1}{\delta}-\left(\frac{1}{\delta}+T-t_w\right)e^{-\delta(T-t_w)}\right)+c_1D\left\{T-t_w-\frac{1}{\delta}\left(1-e^{-\delta(T-t_w)}\right)\right\}$$
$$\left.+c\left\{W\left(1-e^{\alpha(t_d-t_r)}\right)+D(t_d-t_w)-D\left(\frac{1}{\alpha}\left(1-e^{\alpha(t_w-t_r)}\right)+\frac{1}{\beta}\left(1-e^{\beta(t_r-t_d)}\right)\right)\right\}\right]$$
$$+\frac{D}{T}\left[-s(T-t_w)e^{-\delta(T-t_w)}+c_1\left(1-e^{-\delta(T-t_w)}\right)\right]=0 \tag{65.b}$$

$$\frac{\partial TC_{L2}(t_r,T)}{\partial t_r}=\frac{1}{T}\left[FDt_r+H\left\{W+D(t_d-2t_r)+Dt_r+\frac{D}{\alpha}\left(e^{\alpha(t_w-t_d)}-1\right)X_2\right\}\right.$$
$$\left.+D\left\{-s(T-t_w)e^{-\delta(T-t_w)}+c_1\left(e^{-\delta(T-t_w)}-1\right)+c\left(e^{\alpha(t_w-t_d)}-1\right)\right\}X_2\right]=0 \tag{65.c}$$

and

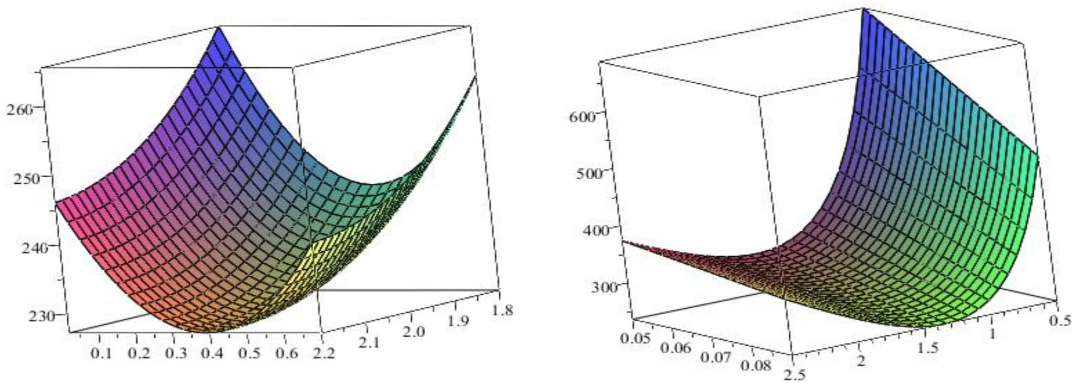
$$\frac{\partial TC_{L2}(t_r,T)}{\partial T}=\frac{1}{T}\left\{sD(T-t_w)e^{-\delta(T-t_w)}+c_1D\left(1-e^{-\delta(T-t_w)}\right)-CD\right\}-\frac{1}{T^2}\left[A+\frac{FDt_r^2}{2}+H\left\{Wt_r+Dt_r(t_d-t_r)\right.\right.$$
$$\left.-\frac{D}{2}\left(t_d^2-t_r^2\right)+\frac{D}{\alpha}\left\{\frac{1}{\alpha}\left(e^{\alpha(t_w-t_d)}-1\right)-(t_w-t_d)\right\}\right\}+\frac{sD}{\delta}\left\{\frac{1}{\delta}-\left(\frac{1}{\delta}+T-t_w\right)e^{-\delta(T-t_w)}\right\}$$
$$\left.+c_1D\left\{T-t_w-\frac{1}{\delta}\left(1-e^{-\delta(T-t_w)}\right)\right\}+cD\left\{\frac{1}{\alpha}\left(e^{\alpha(t_w-t_d)}-1\right)-(t_w-t_d)\right\}\right]=0 \tag{65.d}$$

Equations [(65.a) and (65.b)] and [(65.c) and (65.d)] can be solved simultaneously for the optimal values of t_{ri} and T_i (say t_{ri}^* and T_i^*) for $i=1, 2$ provided, it also satisfies the following sufficient conditions

$$\frac{\partial^2 TC_{Li}(t_r,T)}{\partial t_r^2}>0,\frac{\partial^2 TC_{Li}(t_r,T)}{\partial T^2}>0\ and\ \left[\left(\frac{\partial^2 TC_{Li}}{\partial t_r^{\,2}}\times\frac{\partial^2 TC_{Li}}{\partial T^2}\right)-\left(\frac{\partial^2 TC_{Li}}{\partial T\partial t_r}\times\frac{\partial^2 TC_{Li}}{\partial t_r\partial T}\right)\right]>0\ for\ i=1,2$$

Details are provided in Appendix see Appendix2.

Mathematically, it is very difficult to prove the sufficient conditions, so convexities of cost function for both cases are shown graphically in figure 6.



Case 1: When $t_d < t_r$ **Case2: When $t_d \geq t_r$**

Figure 6: Convexity of cost function for LIFO w.r.t. t_r and T.

4. SOLUTION PROCEDURE

The procedure for finding the economic ordering policy is as follows:

Step 1: Determine t_{r1}^*, T_1^* from equation (31.a) and (31.b), if $t_d < t_w$, substituting these values in the equations (13), (16) and (17) the optimal values of S_f^*, Q_f^* *and* TC_{f1} (t_r^*, T^*) can be obtained respectively.

Step 2: Determine t_{r2}^*, T_2^* from equation (31.c), (31.d), if $t_d > t_w$, substituting these values in the equations (26), (28) and (30) the optimal values of S_f^*, Q_f^* *and* TC_{f2} (t_r^*, T^*) can be obtained respectively.

Step 3: By comparing TC_{f1} (t_r^*, T^*) and TC_{f2} (t_r^*, T^*), select the order size and cycle length with the least total system cost evaluated in Step1 and Step 2.

Step 4: Determine t_{r1}^*, T_1^* from equation (65.a) and (65.b), if $t_d < t_r$, substituting these values in the equations (44), (47) and (48) the optimal values of S_L^*, Q_L^* *and* TC_{L1} (t_r^*, T^*) can be obtained respectively.

Step 5: Determine t_{r2}^*, T_2^* from equation (65.c), (65.d), if $t_d > t_r$, substituting these values in the equations (60), (62) and (63) the optimal values of S_L^*, Q_L^* *and* TC_{L2} (t_r^*, T^*) can be obtained respectively.

Step 6: By comparing TC_{L1} (t_r^*, T^*) and TC_{L2} (t_r^*, T^*), select the order size and cycle length with the least total system cost evaluated in Step4 and Step 5.

Step 7: Now compare the total optimal cost for both policies (FIFO/LIFO) i.e. TC_{fi} (t_r^*, T^*) and TC_{Li} (t_r^*, T^*). The policy having minimum total optimal cost is selected.

5. SPECIAL CASES

Case1. When $t_d = 0$ i.e. instantaneous deterioration

Sub Case 1. In this case the total cost function for FIFO policy is given by equations (30) reduces to

$$TC_f(t_r,T)=\frac{1}{T}\left[A+\frac{F}{\beta}\left\{\left(S_f-W\right)\left(1-e^{-\beta t_w}\right)+D\left\{\frac{1}{\beta}\left(e^{\beta(t_r-t_w)}-1\right)-\left(t_r-t_w\right)\right\}\right\}\right.$$
$$\left.+\frac{sD}{\delta}\left\{\frac{1}{\delta}-\left(\frac{1}{\delta}+T-t_r\right)e^{-\delta(T-t_r)}\right\}+c_1D\left\{T-t_r-\frac{1}{\delta}\left(1-e^{-\delta(T-t_r)}\right)\right\}+C\left(Q-DT\right)\right]$$

Sub Case 2. In this case the total cost function for LIFO policy is given by equations (63) reduces to

$$TC_L(t_r,T)=\frac{1}{T}\left[A+F\frac{D}{\beta}\left(\frac{1}{\beta}\left(e^{\beta t_r}-1\right)-t_r\right)+H\left\{\frac{W}{\alpha}\left(1-e^{-\alpha t_r}\right)+\frac{D}{\alpha}\left(\frac{1}{\alpha}\left(e^{\alpha(t_w-t_r)}-1\right)-\left(t_w-t_r\right)\right)\right\}\right.$$
$$\left.+s\frac{D}{\delta}\left(\frac{1}{\delta}-\left(\frac{1}{\delta}+T-t_w\right)e^{-\delta(T-t_w)}\right)+c_1D\left\{T-t_w-\frac{1}{\delta}\left(1-e^{-\delta(T-t_w)}\right)\right\}+C\left(Q_L-DT\right)\right]$$

6. NUMERICAL AND SENSITIVITY ANALYSIS

To illustrate the results, let us consider an inventory system with the following data: A=\$ 250per cycle, c = \$ 10/unit, s = \$ 5/unit/year, c_1 = \$ 5/unit/year, H = \$ 0.5/unit/year, F = \$ 0.7/unit/year, W = 200 units, D = 300 units/year, $\alpha = 0.05$, $\beta = 0.03$units/year.

Using the proposed solution procedure the results are as follows:
For FIFO Model,
t_w = 0.658 year, t_r = 1.206 year, T = 1.345 year, S_F = 368.496 units, Q_F = 407.748 units, TC_F = \$ 360.854 and deterioration cost=\$ 67.858
For LIFO Model,
t_w = 1.197 year, t_r = 0.559 year, T = 1.337 year, S_L = 366.072 units, Q_L = 409.413 units, TC_F = \$ 365.648 and deterioration cost=\$ 88.505

As the total cost in FIFO policy is less than of LIFO policy, thus FIFO dispatching policy is preferred over LIFO.
Further, the sensitivity analysis on major parameters t_d, α, β, δ and holding costs (*H and F*) have been discussed and shown in table 1-5 respectively.

a) In order to study the effect of non deteriorating period i.e. t_d on the policy, we consider the different values of t_d and results are summarized in Table 1:

t_d	FIFO							LIFO							Policy Selecte
	t_w	t_r	T	S_f	Q_f	D.(	TC_F	t_r	t_w	T	S_L	Q_L	D.C	TC_L	
0	0.6	1.2	1.3	370.	413.0	79.	369.4	0.5	1.2	1.3	370.4	410.8	100.7	381.45	FIFO
0.0	0.6	1.2	1.3	368.	407.8	67.	360.8	0.5	1.1	1.3	366.0	409.4	88.50	365.64	FIFO
0.2	0.6	1.2	1.3	366.	403.7	46.	340.7	0.5	1.1	1.3	364.5	401.8	65.13	344.21	FIFO
0.5	0.6	1.2	1.3	369.	405.0	24.	326.8	0.5	1.2	1.3	363.2	398.0	36.95	321.84	LIFO
0.7	0.6	1.2	1.3	380.	414.9	12.	315.6	0.5	1.2	1.3	377.4	410.3	19.05	304.23	LIFO
1	0.6	1.3	1.4	399.	432.7	8.2	310.8	0.6	1.3	1.4	399.3	430.9	8.139	292.39	LIFO

Table 1: Effect of non deteriorating period (t_d) on the policy selection

Following observations can be made from Table 1:

- As t_d (time during which no deterioration occurs) increases, cycle length (T) and cost of deteriorating units decreases which eventually results in a decrease in the total average cost.
- When $t_d = 0$ i.e. deterioration starts at the beginning of the cycle (instantaneous deterioration) the deterioration cost increases in the case of instantaneous deteriorating items as compared to the non instantaneous items as the number of deteriorating units are more. Hence non instantaneous deteriorating units are very much useful in reducing the deterioration cost as it reduces the Total cost. Also the Deterioration cost is higher in LIFO policy than that of FIFO policy, so it is obvious that the decision maker will adopt FIFO policy rather than LIFO policy.

b) In order to study the effect of deterioration on the policy selection, holding costs in both the warehouses are assumed to be equal ($H = F = 0.5$) and the proposed model has been demonstrated under two situations. Firstly it is assumed that the deterioration rate in OW is greater than that of the RW and vice-versa.

$r\left(\alpha/\beta\right)$	Q_F	TC_F	Q_L	TC_L	**Policy Suggested**
0.1	438.746	317.837	470.696	290.642	LIFO
0.5	445.287	322.186	462.957	307.252	LIFO
1	453.271	327.488	453.271	327.488	EITHER
2	468.642	337.671	433.352	366.243	FIFO
4	497.270	356.549	386.837	436.858	FIFO

Table 2: Effect of deterioration rates on the policy selection

Based on the results as shown in Tables 2, we obtain the following managerial insights:

- When deterioration rate in OW is greater than that of RW i.e. $(\alpha > \beta)$ then $TC_F < TC_L$, thus it is optimal to evacuate OW earlier to RW in order to maintain freshness of goods which result in lesser deterioration cost. Hence in that circumstances decision maker would prefer FIFO dispatch policy than LIFO.
- When deterioration rate in RW is greater than that of OW i.e. $(\alpha < \beta)$ then $TC_F > TC_L$ as the units in RW deteriorate more rapidly so keeping stock in RW for longer period would result in more deterioration cost. Hence the decision maker would use LIFO dispatch policy rather than FIFO.
- When $\alpha = \beta$, $TC_F = TC_L$ both the policies acquiesces identical results, hence the decision maker may use either FIFO or LIFO dispatch policy.

c) In order to study the effect of holding cost on the policy selection, by taking different combinations of *H* and *F*, and the proposed model has been demonstrated under two situations. Firstly it is assumed that the deterioration rate in OW is greater than that of the RW and vice-versa.

H	F	TC_F	TC_L	**Policy Suggested**
0.5	0.5	334.336	353.576	FIFO
	0.6	348.490	357.902	FIFO
	0.7	360.854	357.648	LIFO
0.6	0.5	338.622	366.916	FIFO
	0.6	353.063	370.967	FIFO
	0.7	365.706	362.467	LIFO
0.7	0.5	342.856	380.094	FIFO
	0.6	357.579	383.869	FIFO
	0.7	370.496	387.121	FIFO

Table 3: Effect of holding costs on the policy selection
(When deterioration rate in OW is higher)

H	*F*	TC_F	TC_L	**Policy Suggested**
0.5	0.5 0.6 0.7	370.305 379.830 388.076	362.393 365.417 368.115	LIFO LIFO LIFO
0.6	0.5 0.6 0.7	375.431 385.220 393.726	376.060 380.882 382.395	FIFO LIFO LIFO
0.7	0.5 0.6 0.7	380.491 388.538 399.300	387.566 390.186 392.515	FIFO FIFO LIFO

Table 4: Effect of holding costs on the policy selection
(When deterioration rate in RW is higher)

Based on the computational results as shown in Tables 3 and Table 4, we obtain the following managerial insights:

- When the holding cost and the deterioration rate both are greater in OW than that of RW, then in this case FIFO policy is recommended; since it will be beneficial for the decision maker to evacuate OW first in order to manage the high holding costs of OW.
- Further, if the holding cost in RW is greater than that of OW but the deterioration rate in RW is less than that of OW, then it is unambiguous from the results that the cost associated with LIFO dispatching policy is less than the FIFO dispatching policy. So, LIFO policy is preferred.
- However, when the holding cost in both the warehouses is same but deterioration rate in RW is high than that of OW, then LIFO policy is used. Since, in order to manage the high deterioration cost.
- When $\alpha = \beta, TC_f = TC_L$, both the policies yield same results, hence the decision maker may use either FIFO or LIFO dispatch policy.

d) Now we study the effect of backlogging parameter δ on the policy selection. Sensitivity analysis is performed by changing δ (increasing or decreasing) the backlogging parameter and keeping all other parameters as same.

δ	S_f	Q_f	TC_F	S_L	Q_L	TC_L	**Policy Suggeste**
1.	370.87	4058.01	363.24	370.48	404.69	364.06	FIFO
0.	368.49	407.746	360.85	368.07	407.41	361.64	FIFO
0.	365.37	411.457	357.72	364.91	411.10	358.48	FIFO
0.	361.12	416.715	353.47	360.62	416.33	354.19	FIFO

Table 5: Effect of backlogging rate on the policy selection

- It is observed from table5 that with the decrease in backlogging parameter *δ*, there is a decrease the initial inventory and total average cost. Because increasing backlogging rate implies more of backlogged demand. So it is advisable that when the backlogging rate is more, the organization should order larger quantity in order to satisfy the backlogged demand. As the $TC_f < TC_L$, thus, in this case FIFO policy is suggested.

7. CONCLUSION

The present study establishes the importance of different dispatch policies in a two warehousing environment for non-instantaneous deteriorating items with partial backlogging rate by developing the two different inventory models under LIFO and FIFO dispatching policies. The proposed solution procedure provides the optimal solution as well as optimal dispatch policy. Further, sensitivity analysis of the optimal solution with respect to major parameters is also carried out. Results clearly show that there is considerable improvement in total cost for non-instantaneously deteriorating items compared with instantaneously deteriorating items.

The developed model can be accustomed to manage the inventory of certain non-instantaneously deteriorating items, e.g. food items (dry fruits, food grains etc.), electronic items (refrigerator, television, etc.), and many more and the proposed model can further be extended by including some more realistic features, such as inventory-level-dependent demand, price-dependent demand, inflation and permissible delay in payments etc.

Acknowledgment: The authors deeply appreciate the valuable suggestions of the anonymous reviewers. The first author would like to acknowledge the support of the Research Grant No.: RC/R&D/2014/6820, provided by the University of Delhi, Delhi, India for accomplishing this research. The third author would like to thank University Grant Commission (UGC) for providing the Non-NET fellowship to accomplish the research.

RECEIVED: FEBRUARY 2015
REVISED: JULY 2015

REFERENCES

[1] GHARE, P.M. and SHRADER, G.F. (1963): A model for exponentially decaying inventories. **Journal of Industrial Engineering**, 14, 238–243.

[2] COVERT, R.P. and PHILIP, G.C. (1973): An EOQ model for items with Weibull distribution deterioration. **American Institute of Industrial Engineering Transactions**, 5, 323–326.

[3] RAAFAT, F. (1991): Survey of literature on continuously deteriorating inventory models. **Journal of Operational Research Society**, 42, 27–37.

[4] GOYAL S.K. and GIRI, B.C. (2001): Recent trends in modeling of deteriorating inventory. **European Journal of Operational Research,** 134, 1–16.

[5] WU, K. S., OUYANG, L. Y. and YANG, C. T. (2006): An Optimal Replenishment Policy for Non-instantaneous Deteriorating Items with Stock-dependent Demand and Partial Backlogging. **International Journal of Production Economics**, 101, 369 – 384.

[6] OUYANG, L. Y., WU, K. S. and YANG, C. T. (2006): A study on an inventory model for non-instantaneous deteriorating items with permissible delay in payments. **Computers and Industrial Engineering**, 51, 637–651.

[7] OUYANG, L. Y., WU, K. S. and YANG, C. T. (2008): Retailer's ordering policy for non-instantaneous deteriorating items with quantity discount, stock dependent demand and stochastic backorder rate. **Journal of the Chinese Institute of Industrial Engineer**s, 25, 62-72.

[8] WU, K. S., OUYANG, L. Y. and YANG, C. T. (2009): Coordinating replenishment and pricing policies for non-instantaneous deteriorating items with price-sensitive demand. **International Journal of System Sciences**, 40, 1273–1281.

[9] JAGGI, C.K. and VERMA, P. (2010): An optimal replenishment policy for non-instantaneous deteriorating items with two storage facilities. **International Journal of Services Operations and Informatic**s, 5, 209-230.

[10] CHANG, C. T., TENG, J. T. and GOYAL, S. K. (2010): Optimal replenishment policies for non-instantaneous deteriorating items with stock-dependent demand. **International Journal of Production Economics,** 123, 62–68.

[11] GEETHA, K. V. and UTHAYAKUMAR, R. (2010): Economic design of an inventory policy for non-instantaneous deteriorating items under permissible delay in payments. **Journal of Computational and Applied Mathematics**, 233, 2492–2505.

[12] SONI, H. N. and PATEL, K. A. (2012): Optimal pricing and inventory policies for non-instantaneous deteriorating items with permissible delay in payment: Fuzzy expected value model. International Journal of Industrial Engineering Computations, 3(3), 281–300.

[13] MAIHAMI, R. and KAMALABADI, I. N. (2012a): Joint pricing and inventory control for non-instantaneous deteriorating items with partial backlogging and time and price dependent demands. **International Journal of Production Economics**, 136, 116– 122.

[14] MAIHAMI, R. and KAMALABADI, I. N. (2012b): Joint control of inventory and its pricing for non-instantaneously deteriorating items under permissible delay in payments and partial backlogging. Mathematical and Computer Modelling, 55(5–6), 1722 – 1733.

[15] SHAH, N. H., SONI, H. N. and PATEL, K. A. (2013): Optimizing inventory and marketing policy for non-instantaneous deteriorating items with generalized type deterioration and holding cost rates. **Omega-International Journal of Management Science**, 41, 421 – 430.

[16] DYE, C.Y. (2013): The effect of preservation technology investment on a non-instantaneous deteriorating inventory model. **Omega-International Journal of Management Science**, 41, 872 – 880.
[17] HARTLEY, V. R. (1976): **Operations Research – A Managerial Emphasis**. Good Year, Santa Monica, California.
[18] SARMA, K.V.S. (1987): A deterministic order-level inventory model for deteriorating items with two storage facilities. **European Journal of Operational Research**, 29, 70–72.
[19] GOSWAMI, A. and CHAUDHURI, K.S. (1992): An economic order quantity model for items with two levels of storage for a linear trend in demand, **Journal of Operational Research** Society, 43, 157–167.
[20] PAKKALA, T.P.M. and ACHARY, K.K. (1992): A deterministic inventory model for deteriorating items with two warehouses and finite replenishment rate. **European Journal of Operational Research**, 57, 71–76.
[21] BHUNIA, A.K. and MAITI, M. (1998): A two-warehouse inventory model for deteriorating items with a linear trend in demand and shortages. **Journal of Operational Research Society**, 49, 287–292.
[22] JAGGI, C.K. and VERMA, P. (2010): A deterministic order level inventory model for deteriorating items with two storage facilities under FIFO dispatching policy. **International Journal of Procurement Management**, 3, 265–278.
[23] ZHOU, Y.W. and YANG, S.L. (2003): A two-warehouse inventory model for items with stock-level-dependent demand rate. **International Journal of Production Economics**, 95, 215–228.
[24] CHUNG, K. J. and HUANG, T. S. (2007): The optimal retailer's ordering policies for deteriorating items with limited storage capacity under trade credit financing, **International Journal of Production Economics**, 106, 127-145.
[25] DAS, B., MAITY, K. and MAITI, M. (2007): A two warehouse supply chain model under possibility/ necessity/ credibility measures. **Mathematical and Computer Modelling**, 46, 398-409.
[26] DYE, C. Y., OUYANG, L. Y. and HSIEH, T. P. (2007): Deterministic inventory model for deteriorating items with capacity constraint and time-proportional backlogging rate. **European Journal of Operational Research**, 178, 789-807.
[27] NIU, B. and XIE, J. (2008): A note on Two-warehouse inventory model with deterioration under FIFO dispatches policy. **European Journal of Operational Research**, 190, 571-577.
[28] RONG, M., MAHAPATRA, N. K. and MAITI, M. (2008): A two warehouse inventory model for a deteriorating item with partially/fully backlogged shortage and fuzzy lead time. **European Journal of Operational Research**, 189, 59-75.
[29] LEE, C. C. (2006): Two-warehouse inventory model with deterioration under FIFO dispatching policy. **European Journal of Operational Research**, 174, 861-873.
[30] LEE, C.C. and HSU, S. L. (2008): A two-warehouse production model for deteriorating inventory items with time-dependent demands. **European Journal of Operational Research**, 194 , 700-710.

Appendix 1

For FIFO Model, the necessary and sufficient conditions for minimizing the total cost are given by

$$\frac{\partial TC_{f1}(t_r,T)}{\partial t_r} = \frac{D}{T}\left[(Ft_d + c\beta)e^{\beta(t_r-t_d)} + \frac{F}{\beta}\left(e^{\beta(t_r-t_d)}-1\right) + \left\{c_1 - s(T-t_r)\right\}e^{-\delta(T-t_r)} - c_1\right] = 0$$

$$\begin{aligned}\frac{\partial TC_{f1}(t_r,\, T)}{\partial T} = &-\frac{1}{T^2}\left[\frac{FD}{\beta}\left\{\left(e^{\beta(t_r-t_d)} - e^{\beta(t_w-t_d)}\right)\left\{t_d + \frac{1}{\beta}\left(1-e^{\beta(t_d-t_w)}\right)\right\} + \left\{\frac{1}{\beta}\left(e^{\beta(t_r-t_w)}-1\right) - (t_r - t_w)\right\}\right\}\right. \\ &+ H\left\{Wt_d - \frac{Dt_d^2}{2} - \frac{D}{\alpha}\left\{\frac{1}{\alpha}\left(1-e^{\alpha(t_w-t_d)}\right) + (t_w - t_d)\right\}\right\} + \frac{sD}{\delta}\left\{\frac{1}{\delta} - \left(\frac{1}{\delta} + T - t_r\right)e^{-\delta(T-t_r)}\right\} \\ &+ c_1 D\left\{T - t_r - \frac{1}{\delta}\left\{1-e^{-\delta(T-t_r)}\right\}\right\} + cD\left\{\frac{1}{\alpha}\left(1-e^{\alpha(t_w-t_d)}\right) + (t_w - t_d) + \frac{1}{\beta}\left(e^{\beta(t_r-t_d)} - e^{\beta(t_w-t_d)}\right)\left(1-e^{\beta(t_d-t_w)}\right)\right. \\ &\left.\left.+\left(\frac{1}{\beta}\left(e^{\beta(t_r-t_w)}-1\right) - (t_r - t_w)\right)\right\}\right] + \frac{D}{T}\left\{s(T-t_r)e^{-\delta(T-t_r)} + c_1\left(1-e^{-\delta(T-t_r)}\right)\right\} = 0\end{aligned}$$

$$\frac{\partial^2 TC_{f1}(t_r,\ T)}{\partial t_r^2} = \frac{D}{T}\left[\left\{F(\beta t_d+1)+c\beta^2\right\}e^{\beta(t_r-t_d)}+\left\{s(1-\delta(T-t_r))+c_1\delta\right\}e^{-\delta(T-t_r)}\right]$$

$$\frac{\partial^2 TC_{f1}(t_r,\ T)}{\partial T^2} = \frac{2}{T^3}\left[\frac{FD}{\beta}\left\{\left(e^{\beta(t_r-t_d)}-e^{\beta(t_w-t_d)}\right)\left\{t_d+\frac{1}{\beta}\left(1-e^{\beta(t_d-t_w)}\right)\right\}+\left\{\frac{1}{\beta}\left(e^{\beta(t_r-t_w)}-1\right)-(t_r-t_w)\right\}\right\}\right.$$
$$+H\left\{Wt_d-\frac{Dt_d^2}{2}-\frac{D}{\alpha}\left\{\frac{1}{\alpha}\left(1-e^{\alpha(t_w-t_d)}\right)+(t_w-t_d)\right\}\right\}+\frac{sD}{\delta}\left\{\frac{1}{\delta}-\left(\frac{1}{\delta}+T-t_r\right)e^{-\delta(T-t_r)}\right\}$$
$$+c_1D\left\{T-t_r-\frac{1}{\delta}\left\{1-e^{-\delta(T-t_r)}\right\}\right\}+cD\left\{\frac{1}{\alpha}\left(1-e^{\alpha(t_w-t_d)}\right)+(t_w-t_d)+\frac{1}{\beta}\left(e^{\beta(t_r-t_d)}-e^{\beta(t_w-t_d)}\right)\left(1-e^{\beta(t_d-t_w)}\right)\right.$$
$$\left.\left.+\left(\frac{1}{\beta}\left(e^{\beta(t_r-t_w)}-1\right)-(t_r-t_w)\right)\right\}\right]-\frac{2D}{T^2}\left\{s(T-t_r)e^{-\delta(T-t_r)}+c_1\left(1-e^{-\delta(T-t_r)}\right)\right\}+\frac{D}{T}\left\{s(1-\delta(T-t_r))+c_1\right\}e^{-\delta(T-t_r)}$$

$$\frac{\partial^2 TC_{f1}(t_r,\ T)}{\partial T\partial t_r} = \frac{\partial^2 TC_{f1}(t_r,\ T)}{\partial t_r\partial T} = -\frac{D}{T^2}\left[(Ft_d+c\beta)e^{\beta(t_r-t_d)}+\frac{F}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)+\left\{c_1-s(T-t_r)\right\}e^{-\delta(T-t_r)}-c_1\right]$$
$$-\frac{D}{T}\left\{s+\delta\left\{c_1-s(T-t_r)\right\}\right\}e^{-\delta(T-t_r)}$$

(A. 1)

$$\frac{\partial TC_{f2}(t_r,T)}{\partial t_r} = \frac{D}{T}\left[F\left\{e^{\beta(t_r-t_d)}t_d+\frac{1}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)\right\}-\left\{c_1+s(T-t_r)\right\}e^{-\delta(T-t_r)}+c_1+c\left(e^{\beta(t_r-t_d)}-1\right)\right]=0$$

$$\frac{\partial TC_{f2}(t_r,\ T)}{\partial T} = -\frac{1}{T^2}\left[A+\frac{HDt_w^2}{2}+FD\left\{\left\{\frac{1}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)+(t_d-t_w)\right\}t_d+\left\{\frac{1}{\beta^2}\left(e^{\beta(t_r-t_d)}-1\right)-\frac{(t_d-t_w)^2}{2}-\frac{(t_r-t_d)}{\beta}\right\}\right\}\right.$$
$$\left.+\frac{sD}{\delta}\left\{\frac{1}{\delta}-\left(\frac{1}{\delta}+T-t_r\right)e^{-\delta(T-t_r)}\right\}+c_1D\left\{T-t_r-\frac{1}{\delta}\left(1-e^{-\delta(T-t_r)}\right)\right\}+cD\left\{\frac{1}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)-(t_r-t_d)\right\}\right]$$
$$+\frac{D}{T}\left[s(T-t_r)e^{-\delta(T-t_r)}+c_1\left(1-e^{-\delta(T-t_r)}\right)\right]=0$$

$$\frac{\partial^2 TC_{f2}(t_r,\ T)}{\partial t_r^2} = \frac{D}{T}\left[F\left\{1+\beta t_d\right\}e^{\beta(t_r-t_d)}+\left\{1+\delta\left(c_1+s(T-t_r)\right)\right\}e^{-\delta(T-t_r)}+c\beta e^{\beta(t_r-t_d)}\right]$$

$$\frac{\partial^2 TC_{f2}(t_r,T)}{\partial T^2} = \frac{2}{T^3}\left[A+\frac{HDt_w^2}{2}+FD\left\{\left\{\frac{1}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)+(t_d-t_w)\right\}t_d+\left\{\frac{1}{\beta^2}\left(e^{\beta(t_r-t_d)}-1\right)-\frac{(t_d-t_w)^2}{2}-\frac{(t_r-t_d)}{\beta}\right\}\right\}\right.$$
$$\left.+\frac{sD}{\delta}\left\{\frac{1}{\delta}-\left(\frac{1}{\delta}+T-t_r\right)e^{-\delta(T-t_r)}\right\}+c_1D\left\{T-t_r-\frac{1}{\delta}\left(1-e^{-\delta(T-t_r)}\right)\right\}+cD\left\{\frac{1}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)-(t_r-t_d)\right\}\right]$$
$$-\frac{2D}{T^2}\left[s(T-t_r)e^{-\delta(T-t_r)}+c_1\left(1-e^{-\delta(T-t_r)}\right)\right]+\frac{D}{T}\left\{1-s\delta(T-t_r)+c_1\right\}e^{-\delta(T-t_r)}$$

$$\frac{\partial^2 TC_{f2}(t_r,\ T)}{\partial T\partial t_r} = \frac{\partial^2 TC_{f1}(t_r,\ T)}{\partial t_r\partial T} = -\frac{D}{T^2}\left[F\left\{e^{\beta(t_r-t_d)}t_d+\frac{1}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)\right\}-\left\{c_1+s(T-t_r)\right\}e^{-\delta(T-t_r)}+c_1+c\left(e^{\beta(t_r-t_d)}-1\right)\right]$$
$$+\frac{D}{T}\left\{\delta\left(c_1+s(T-t_r)\right)+s\right\}e^{-\delta(T-t_r)}$$

(A.2)

Appendix 2

For LIFO Model, the necessary and sufficient conditions for minimizing the total cost are given by

$$\frac{\partial TC_{L1}(t_r,T)}{\partial t_r} = \frac{1}{T}\left[FDe^{\beta(t_r-t_d)}\left(t_d+\frac{1}{\beta}\right)+H\left\{We^{\alpha(t_d-t_r)}+\frac{D}{\alpha}\left(1-e^{\alpha(t_w-t_r)}\right)(1-X_1)\right\}+D\left\{\left(c_1-\frac{2s}{\delta}-s(T-t_w)\right)e^{-\delta(T-t_w)}-c_1\right\}X_1\right.$$
$$\left.+c\left\{\alpha We^{\alpha(t_d-t_r)}-DX_1-D\left(e^{\alpha(t_w-t_r)}(X_1-1)-e^{\beta(t_r-t_d)}\right)\right\}\right]=0$$

$$\frac{\partial TC_{L1}(t_r,\ T)}{\partial T} = -\frac{1}{T^2}\left[A+F\left\{\frac{D}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)\left(t_d+\frac{1}{\beta}\right)+\frac{Dt_d^2}{2}-\frac{D}{\beta}(t_r-t_d)\right\}+H\left\{Wt_d+\frac{W}{\alpha}\left(1-e^{\alpha(t_d-t_r)}\right)\right.\right.$$
$$\left.+\frac{D}{\alpha}\left(t_r-t_w-\frac{1}{\alpha}\left(1-e^{\alpha(t_w-t_r)}\right)\right)\right\}+s\frac{D}{\delta}\left(\frac{1}{\delta}-\left(\frac{1}{\delta}+T-t_w\right)e^{-\delta(T-t_w)}\right)+c_1D\left\{T-t_w-\frac{1}{\delta}\left(1-e^{-\delta(T-t_w)}\right)\right\}$$
$$\left.+c\left\{W\left(1-e^{\alpha(t_d-t_r)}\right)+D(t_d-t_w)-D\left(\frac{1}{\alpha}\left(1-e^{\alpha(t_w-t_r)}\right)+\frac{1}{\beta}\left(1-e^{\beta(t_r-t_d)}\right)\right)\right\}\right]$$
$$+\frac{D}{T}\left[-s(T-t_w)e^{-\delta(T-t_w)}+c_1\left(1-e^{-\delta(T-t_w)}\right)\right]=0$$

$$
\begin{aligned}
\frac{\partial^2 TC_{L1}(t_r,T)}{\partial t_r^2} &= \frac{1}{T}\Bigg[FD\beta e^{\beta(t_r-t_d)}\left(t_d+\frac{1}{\beta}\right)+H\left\{\alpha W e^{\alpha(t_d-t_r)}-\alpha D e^{\alpha(t_w-t_r)}\left(X_1-1\right)^2-\frac{D}{\alpha}\left(1-e^{\alpha(t_w-t_r)}\right)Y_1\right\} \\
&\quad +D\left\{\left(c_1-\frac{2s}{\delta}-s(T-t_w)\right)e^{-\delta(T-t_w)}-c_1\right\}Y_1+c\left\{-\alpha^2 W e^{\alpha(t_d-t_r)}-DY_1-D\left(e^{\alpha(t_w-t_r)}(Y_1-1)+\alpha(X_1-1)-\beta e^{\beta(t_r-t_d)}\right)\right\}\Bigg] \\
\frac{\partial^2 TC_{L1}(t_r,T)}{\partial T^2} &= \frac{2}{T^3}\Bigg[A+F\left\{\frac{D}{\beta}\left(e^{\beta(t_r-t_d)}-1\right)\left(t_d+\frac{1}{\beta}\right)+\frac{Dt_d^2}{2}-\frac{D}{\beta}(t_r-t_d)\right\}+H\Bigg\{Wt_d+\frac{W}{\alpha}\left(1-e^{\alpha(t_d-t_r)}\right) \\
&\quad +\frac{D}{\alpha}\left(t_r-t_w-\frac{1}{\alpha}\left(1-e^{\alpha(t_w-t_r)}\right)\right)\Bigg\}+s\frac{D}{\delta}\left(\frac{1}{\delta}-\left(\frac{1}{\delta}+T-t_w\right)e^{-\delta(T-t_w)}\right)+c_1D\left\{T-t_w-\frac{1}{\delta}\left(1-e^{-\delta(T-t_w)}\right)\right\} \\
&\quad +c\left\{W\left(1-e^{\alpha(t_d-t_r)}\right)+D(t_d-t_w)-D\left(\frac{1}{\alpha}\left(1-e^{\alpha(t_w-t_r)}\right)+\frac{1}{\beta}\left(1-e^{\beta(t_r-t_d)}\right)\right)\right\}\Bigg] \\
&\quad -\frac{2D}{T^2}\left[-s(T-t_w)e^{-\delta(T-t_w)}+c_1\left(1-e^{-\delta(T-t_w)}\right)\right]+\frac{D}{T}\left[\delta s(T-t_w)+s+\delta c_1\right]e^{-\delta(T-t_w)} \\
\frac{\partial^2 TC_{L1}(t_r,T)}{\partial T\partial t_r} &= \frac{\partial^2 TC_{L1}(t_r,T)}{\partial t_r\partial T}=-\frac{1}{T^2}\Bigg[FDe^{\beta(t_r-t_d)}\left(t_d+\frac{1}{\beta}\right)+H\left\{We^{\alpha(t_d-t_r)}+\frac{D}{\alpha}\left(1-e^{\alpha(t_w-t_r)}\right)(1-X_1)\right\} \\
&\quad +D\left\{\left(c_1-\frac{2s}{\delta}-s(T-t_w)\right)e^{-\delta(T-t_w)}-c_1\right\}X_1+c\left\{\alpha W e^{\alpha(t_d-t_r)}-DX_1-D\left(e^{\alpha(t_w-t_r)}(X_1-1)-e^{\beta(t_r-t_d)}\right)\right\}\Bigg] \\
&\quad -\frac{D}{T}\left\{s+\delta\left(c_1-\frac{2s}{\delta}-s(T-t_w)\right)\right\}e^{-\delta(T-t_w)}X_1
\end{aligned}
\tag{B.1}
$$

$$
\begin{aligned}
\frac{\partial TC_{L2}(t_r,T)}{\partial t_r} &= \frac{1}{T}\Bigg[FDt_r+H\left\{W+D(t_d-2t_r)+Dt_r+\frac{D}{\alpha}\left(e^{\alpha(t_w-t_d)}-1\right)X_2\right\} \\
&\quad +D\left\{-s(T-t_w)e^{-\delta(T-t_w)}+c_1\left(e^{-\delta(T-t_w)}-1\right)+c\left(e^{\alpha(t_w-t_d)}-1\right)\right\}X_2\Bigg]=0 \\
\frac{\partial TC_{L2}(t_r,T)}{\partial T} &= \frac{1}{T}\left\{sD(T-t_w)e^{-\delta(T-t_w)}+c_1D\left(1-e^{-\delta(T-t_w)}\right)-CD\right\}-\frac{1}{T^2}\Bigg[A+\frac{FDt_r^2}{2}+H\Bigg\{Wt_r+Dt_r(t_d-t_r) \\
&\quad -\frac{D}{2}\left(t_d^2-t_r^2\right)+\frac{D}{\alpha}\left\{\frac{1}{\alpha}\left(e^{\alpha(t_w-t_d)}-1\right)-(t_w-t_d)\right\}\Bigg\}+\frac{sD}{\delta}\left\{\frac{1}{\delta}-\left(\frac{1}{\delta}+T-t_w\right)e^{-\delta(T-t_w)}\right\} \\
&\quad +c_1D\left\{T-t_w-\frac{1}{\delta}\left(1-e^{-\delta(T-t_w)}\right)\right\}+cD\left\{\frac{1}{\alpha}\left(e^{\alpha(t_w-t_d)}-1\right)-(t_w-t_d)\right\}\Bigg]=0 \\
\frac{\partial^2 TC_{L2}(t_r,T)}{\partial t_r^2} &= \frac{1}{T}\Bigg[FD+H\left\{W-D+\frac{D}{\alpha}\left(e^{\alpha(t_w-t_d)}-1\right)Y_2+De^{\alpha(t_w-t_d)}X_2\right\}+D\Bigg\{-s(T-t_w)e^{-\delta(T-t_w)}+c_1\left(e^{-\delta(T-t_w)}-1\right) \\
&\quad +c\left(e^{\alpha(t_w-t_d)}-1\right)\Bigg\}Y_2+D\left\{\left\{1+\delta(T-t_w)+\delta c_1\right\}se^{-\delta(T-t_w)}+\alpha ce^{\alpha(t_w-t_d)}\right\}(X_2)^2\Bigg] \\
\frac{\partial^2 TC_{L2}(t_r,T)}{\partial T^2} &= -\frac{1}{T^2}\left\{sD(T-t_w)e^{-\delta(T-t_w)}+c_1D\left(1-e^{-\delta(T-t_w)}\right)-CD\right\}-\frac{1}{T^2}\Bigg[A+\frac{FDt_r^2}{2}+H\Bigg\{Wt_r+Dt_r(t_d-t_r) \\
&\quad -\frac{D}{2}\left(t_d^2-t_r^2\right)+\frac{D}{\alpha}\left\{\frac{1}{\alpha}\left(e^{\alpha(t_w-t_d)}-1\right)-(t_w-t_d)\right\}\Bigg\}+\frac{sD}{\delta}\left\{\frac{1}{\delta}-\left(\frac{1}{\delta}+T-t_w\right)e^{-\delta(T-t_w)}\right\} \\
&\quad +c_1D\left\{T-t_w-\frac{1}{\delta}\left(1-e^{-\delta(T-t_w)}\right)\right\}+cD\left\{\frac{1}{\alpha}\left(e^{\alpha(t_w-t_d)}-1\right)-(t_w-t_d)\right\}\Bigg] \\
&\quad +\frac{1}{T}\left[\frac{sD}{\delta}\left\{\frac{1}{\delta}+\delta\left(\frac{1}{\delta}+T-t_w\right)e^{-\delta(T-t_w)}+e^{-\delta(T-t_w)}\right\}+c_1D\left(1-e^{-\delta(T-t_w)}\right)\right] \\
\frac{\partial^2 TC_{L2}(t_r,T)}{\partial T\partial t_r} &= \frac{\partial^2 TC_{L2}(t_r,T)}{\partial t_r\partial T}=-\frac{1}{T^2}\Bigg[FDt_r+H\left\{W+D(t_d-2t_r)+Dt_r+\frac{D}{\alpha}\left(e^{\alpha(t_w-t_d)}-1\right)X_2\right\} \\
&\quad +D\left\{-s(T-t_w)e^{-\delta(T-t_w)}+c_1\left(e^{-\delta(T-t_w)}-1\right)+c\left(e^{\alpha(t_w-t_d)}-1\right)\right\}X_2\Bigg]+\frac{D}{T}\left\{\delta s(T-t_w)-s-\delta c_1\right\}e^{-\delta(T-t_w)}X_2
\end{aligned}
\tag{B.2}
$$

Where $X_1=1-\alpha We^{\alpha(t_d-t_r)}\left(1+\frac{\alpha W}{D}e^{\alpha(t_d-t_r)}\right)$ and $X_2=\frac{D}{D+\alpha\left(W+D(t_r-t_d)\right)}$

$Y_1=\alpha^2 We^{\alpha(t_d-t_r)}\left(1+\frac{2\alpha W}{D}e^{\alpha(t_d-t_r)}\right)$ and $Y_2=\frac{-D^2}{\left\{D+\alpha\left(W+D(t_r-t_d)\right)\right\}^2}$