A TUTORIAL NOTE ON MULTIOBJECTIVE SEMI-INFINITE PROGRAMMING

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ABSTRACT
This tutorial note deals with the class MOSIP of multiobjective semi-infinite programming problems which are defined by finitely many objective functions and infinitely many inequality constraints in a finite-dimensional space. This note overviews recent results on solution types, constraint qualifications and optimality conditions for nonconves problems of the type MOSIP. Moreover, under additional convexity assumptions, weak, strong and converse duality results are reviewed with respect to both efficiency and weak efficiency.
KEYWORDS: semi-infinite programming; multiobjective programming; duality; (non-)convex case; (locally) properly efficient points
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RESUMEN
Los problemas mutiobjetivos de la programación semi-definida (MOSIP por sus cifras en inglés) tratan la miinimización de una cantidad finita de funciones en un subconjunto de un espacio de dimensión finita que está descrito por una cantidad infinita de restricciones de desigualdad. Esta nota presenta un resumen de los resultados más recientes con respecto a tipos de soluciones, condiciones de regularidad del conjunto de soluciones factibles y condiciones de optimalidad en el caso no convexo. Se revisa la dualidad débil, fuerte e inversa desde el punto de vista de la eficiencia y la eficienfcia débil en el caso convexo.

1. INTRODUCTION

In this tutorial note we consider nonlinear and in general nonconvex multiobjective semi-infinite programming problems. These are mathematical programming problems in a *finite-dimensional* space with finitely many objective functions and *infinitely* many inequality constraints. Note that further finitely many equality constraints could be included straightforwardly but we omitted them in order to avoid unnecessary technicalities.

The problem class under consideration is given by the multiobjective semi-infinite programming problem MOSIP as follows:

$$\text{MOSIP} \quad \min f(x) \quad \text{s.t.} \quad x \in M$$

with the vector $f = (f_1, \dots, f_q)^\top$ of objective functions $f_i \in C^2(\mathbb{R}^n, \mathbb{R})$, $i = 1, \dots, q$ (as usual, $C^k(\mathbb{R}^n, \mathbb{R})$ denotes the space of k-times continuously differentiable real-valued functions defined on $\mathbb{R}^n$) as well as the feasible set

$$M = \{x \in \mathbb{R}^n \mid g(x, y) \leq 0,\ \ y \in Y\}.$$

Here, $Y \subset \mathbb{R}^m$ is a compact and in general infinite index set and $g \in C^2(\mathbb{R}^n \times Y, \mathbb{R})$. Each element $y \in Y$ represents a corresponding inequality constraint $g(., y) \leq 0$. Given a feasible point $\overline{x} \in M$, the index set of active inequality constraints at $\overline{x}$ is defined as the compact set

$$Y_0(\overline{x}) = \{y \in Y \mid g(\overline{x}, y) = 0\}$$

and, obviously, each $\overline{y} \in Y_0(\overline{x})$ is a global maximizer of the parameter-dependent function $g(\overline{x}, \cdot)_{|Y}$. The latter means that the feasible set can be described by *one non-differentiable* constraint as

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$$M = \left\{ x \in \mathbb{R}^n \mid \max_{y \in Y} g(\overline{x}, y) \leq 0 \right\}.$$

In this paper we will not discuss this nonsmooth approach to MOSIP.

Both multiobjective and semi-infinite programming have been very active research areas within nonlinear differentiable mathematical programming for several decades including a huge number of applications. We refer to [6, 14, 21, 22, 24] on semi-infinite programming and to the standard book [4] on multiobjective programming. However, there are only a few systematic results on the combination of multiobjective and semi-infinite programming although there are several applications for the model MOSIP; we refer e.g. to simultaneous Chebyshev approximation [3], approximation problems in the petroleum industry [7, 8, 16] or portfolio optimization [24]. In their recent papers [10, 11, 12], the authors tried to fill this gap to some extent by presenting a more systematic approach to the class defined by MOSIP.

The goal of this paper is to present several important results from [10, 11, 12] as an overview in a tutorial form. In particular, this paper is organized as follows. Section 2 contains some auxiliary results and generalizations of solution types from multiobjective programming to problems of the type MOSIP. Section 3 covers constraint qualifications and optimality conditions for MOSIP and Section 4 duality results for convex problems. Finally, some conclusions are given in Section 5. Note that all results presented in this paper hold analogously for *maximize*- (instead of *minimize*-) multiobjective semi-infinite programming problems.

2. AUXILIARY RESULTS AND SOLUTION TYPES

We start with some basic notations. If $h \in C^1(\mathbb{R}^n, \mathbb{R})$, then we denote by the row vector $Dh(\overline{x})$ $(D_{x^1}h(\overline{x}))$ its gradient (partial gradient with respect to the subvector x^1 of x) at $\overline{x} \in \mathbb{R}^n$. If $h \in C^2(\mathbb{R}^n, \mathbb{R})$, then the second derivatives are defined analogously.
Given $c \in \mathbb{R}^n$ denote its components by c_i, $i = 1, \dots, n$ and for $c, e \in \mathbb{R}^n$ let

- $c \leqq e$, if $c_i \leq e_i$, $i = 1, \dots, n$,
- $c < e$, if $c_i < e_i$, $i = 1, \dots, n$,
- $c \leq e$, if $c_i \leq e_i$, $i = 1, \dots, n$ and $c \neq e$.

Denote the origin in $\mathbb{R}^q$ by 0_q and let $\mathbb{R}^q_+ = \{z \in \mathbb{R}^q \mid z \geqq 0_q\}$. The Euclidean norm is presented by $\|\cdot\|$ and for $\overline{x} \in \mathbb{R}^n$ and a real number $\varepsilon > 0$ let $B(\overline{x}, \varepsilon) = \{x \in \mathbb{R}^n \mid \|x - \overline{x}\| < \varepsilon\}$.

Types of solutions for MOSIP

We recall the following terminology for different types of solutions for a multiobjective programming problem which can be straightforwardly applied to MOSIP.

Definition 2.1. (a) A point $\overline{x} \in M$ is called *efficient for MOSIP* if there does not exist any $x \in M$ with $f(x) \leq f(\overline{x})$.
(b) A point $\overline{x} \in M$ is called *weakly efficient for MOSIP* if there does not exist any $x \in M$ with $f(x) < f(\overline{x})$.
(c) A point $\overline{x} \in M$ is called *locally (weakly) efficient for MOSIP* on $B(\overline{x}, \varepsilon)$ if there exists a real number $\varepsilon > 0$ and if there does not exist any $x \in B(\overline{x}, \varepsilon) \cap M$ with $f(x) \leq f(\overline{x})$ $(f(x) < f(\overline{x}))$.

Note that (locally) (weakly) efficient points are sometimes also called Pareto optimal points. In order to further characterize efficient points we recall in the following definition two concepts of proper efficient points which where introduced by *Geoffrion* [5] as well as by *Kuhn* and *Tucker* [18].

Definition 2.2. (See e.g. [4, 5, 18])
(a) A point $\overline{x} \in M$ is called *locally properly efficient* for MOSIP *in the sense of Geoffrion* (shortly: *G-locally properly efficient*) if there exists a real number $\varepsilon > 0$ such that

- $\overline{x}$ is a locally efficient point for MOSIP with respect to $B(\overline{x}, \varepsilon)$ and

• there exists a real number $Q > 0$ such that for each index $i \in \{1, \ldots, q\}$ and any $x \in B(\overline{x}, \varepsilon) \cap M$ with $f_i(x) < f_i(\overline{x})$ there exists an index $j \in \{1, \ldots, q\}$ such that $f_j(x) > f_j(\overline{x})$ and

$$\frac{f_i(\overline{x}) - f_i(x)}{f_j(x) - f_j(\overline{x})} \leq Q.$$

(b) A point $\overline{x} \in M$ is called *locally properly efficient* for MOSIP *in the sense of Kuhn and Tucker* (shortly: *KT-locally properly efficient*) if there exists a real number $\varepsilon > 0$ such that
• $\overline{x}$ is a locally efficient point for MOSIP with respect to $B(\overline{x}, \varepsilon)$ and
• the following system has no solution $d \in \mathbb{R}^n$:

$$\begin{aligned} &Df_i(\overline{x})d \leq 0, \ i = 1, \ldots, q, \\ &Df_k(\overline{x})d < 0, \ \text{for some } k \in \{1, \ldots, q\}, \\ &Dg_j(\overline{x})d \leq 0, \ j \in J_0(\overline{x}). \end{aligned}$$

By substituting $B(\overline{x}, \varepsilon)$ by M in the latter definition (and deleting the word *locally*), the properties of being a *G-properly efficient point* for MOSIP respectively a *KT-properly efficient point* for MOSIP are analogously defined.

If the point $\overline{x} \in M$ is a locally efficient point for MOSIP, then for any index $i \in \{1, \ldots, q\}$ and any $x \in M$ with $f_i(x) < f_i(\overline{x})$ there exists an index $j \in \{1, \ldots, q\}$ with $f_j(x) > f_j(\overline{x})$. If further $\overline{x}$ is a G-properly efficient point for MOSIP, then the ratio between the improvement of one objective function and the decrease of another objective function is bounded by the finite number $Q > 0$ in the previous definition. We also refer to Example 4.2 in [10] as an illustration for the two different recalled concepts of proper efficiency.

Convex problems

We call MOSIP a *convex problem* if f_i, $i = 1, \ldots, q$ are convex functions and $g(\cdot, \overline{y})$ is a convex function for each $\overline{y} \in Y$. Then, the convex feasible set M can be described by using cone constraints (see e.g. [23]). For this consider the space $C(Y)$ of continuous real-valued functions

$$c : Y \to \mathbb{R}$$

and its dual space $C(Y)^*$. For $c \in C(Y)$ and $\mu \in C(Y)^*$ we have the product

$$\mu(c) = \langle \mu, c \rangle := \int_Y c(y) d\mu(y).$$

After defining the map

$$G : x \in \mathbb{R}^n \mapsto g(x, \cdot) \in C(Y)$$

and the cone

$$K = \{c \in C(Y) \mid c(y) \geqslant 0 \text{ for all } y \in Y\}$$

the feasible set of MOSIP can be rewritten as

$$M = \{x \in \mathbb{R}^n \mid G(x) \in -K\}.$$

Define further

$$K^* = \{\mu \in C(Y)^* \mid \langle \mu, c \rangle \geq 0 \text{ for all } c \in K\}.$$

The weighted sum optimization problem

The so-called *weighted sum optimization problem* (assigned to MOSIP) is defined as

$$\min \ \lambda^{\top} f(x) \quad \text{s.t.} \quad x \in M \tag{2.1}$$

with non-negative weights $\lambda \geq 0_q$, $\lambda = (\lambda_1, \ldots, \lambda_q)$. A well-known relationship between solutions of MOSIP and those of (2.1) is summarized in the following.

Lemma 2.1. (cf. e.g. [4]). Let $\overline{x} \in M$.

(a) If $\overline{x}$ is a global minimizer of (2.1) for some $\lambda > 0_q$ ($\lambda \geq 0_q$), then $\overline{x}$ is a (weakly) efficient point for MOSIP.

(b) Suppose that MOSIP is a convex problem. Then, $\overline{x}$ is a G-properly efficient point for MOSIP if and only if $\overline{x}$ is a global minimizer of (2.1) for some $\lambda > 0_q$.

(c) Suppose that MOSIP is a convex problem. Then, $\overline{x}$ is a weakly efficient point for MOSIP if and only if $\overline{x}$ is a global minimizer of (2.1) for some $\lambda \geq 0_q$.

3. CONSTRAINT QUALIFICATIONS AND OPTIMALITY CONDITIONS

In this section we discuss constraint qualifications for MOSIP and optimality conditions with respect to locally weakly efficient as well as G- and KT-properly efficient points. The results are mainly taken from [10].

Cones

For the presentation of constraint qualifications we need the following two cones.

Contingent cone $T(\overline{x}, M) \subseteq \mathbb{R}^n$ of M at $\overline{x} \in M$:

$d \in T(\overline{x}, M)$ if and only if there exist sequences $\{t^{\upsilon}\}_{\upsilon \in \mathbb{N}}$ and $\{d^{\upsilon}\}_{\upsilon \in \mathbb{N}}$ such that $t^{\upsilon} \downarrow 0$, $d^{\upsilon} \to d$ and $\overline{x} + t^{\upsilon} d^{\nu} \in M$ for all $\nu \in \mathbb{N}$.

Cone of attainable directions $A(\overline{x}, M) \subseteq \mathbb{R}^n$ of M at $\overline{x} \in M$ (see [1]):

$$A(\overline{x}, M) = \left\{ d \in \mathbb{R}^n \setminus \{0\} \;\middle|\; \begin{array}{c} \text{There exist some } \tau > 0 \text{ and a continuously} \\ \text{differentiable arc } E : [0, \tau) \to \mathbb{R}^n \text{ such that} \\ E(0) = \overline{x}, \ E'(0) = d, \text{ and } E(t) \in M, \ t \in [0, \tau) \end{array} \right\}.$$

Constraint qualifications

We recall the following four constraint qualifications.

EMFCQ: The *extended Mangasarian-Fromovitz constraint qualification* is said to hold at $\overline{x} \in M$ if the set

$$\{d \in \mathbb{R}^n \mid D_x g(\overline{x}, y)\, d < 0, \ y \in Y_0(\overline{x})\}$$

is non-empty.

EKTCQ: The *extended Kuhn-Tucker constraint qualification* is said to hold at $\overline{x} \in M$ if

$$\{d \in \mathbb{R}^n \mid D_x g(\overline{x}, y)\, d \leq 0, \ y \in Y_0(\overline{x})\} \subseteq \ \text{cl } A(\overline{x}, M).$$

EACQ: The *extended Abadie constraint qualification* is said to hold at $\overline{x} \in M$ if

$$T(\overline{x}, M) = \{d \in \mathbb{R}^n \mid D_x g(\overline{x}, y)\, d \leq 0, \ y \in Y_0(\overline{x})\}.$$

SC: If MOSIP is a convex problem, then MOSIP is said to fulfill the *Slater condition* if there exists a point $\overline{x} \in M$ such that $G(\overline{x}) \in \mathit{int}(-K)$ where *int* denotes the set of interior points (that is, $g(\overline{x}, y) < 0$ for all $y \in Y$).

For a deeper insight and original versions (for finite problems) on these constraint qualifications we refer to [17, 18, 19, 20, 24, 25] as well as to [9, 24] for a detailed discussion on constraint qualifications for (generalized) semi-infinite programming problems.

Since $Y \subset \mathbb{R}^m$ is compact, for $\overline{x} \in M$ the set

$$V(\overline{x}) = \{D_x g(\overline{x}, y) \mid y \in Y_0(\overline{x})\}$$

is also compact. However, the convex cone hull $co(V(\overline{x}))$ of the latter set is not closed in general; it is well-known that for some optimality conditions in semi-infinite programming the closeness of $co(V(\overline{x}))$ plays a crucial role ([9, 10, 14]). The next lemma delivers some properties related to the constraint qualifications presented above.

Lemma 3.2. (a) (see [9]) For $\overline{x} \in M$ the following chain of implications holds:

$$\textit{EMFCQ holds at } \overline{x} \ \rightarrow\ \textit{EKTCQ holds at } \overline{x} \ \rightarrow\ \textit{EACQ holds at } \overline{x}.$$

The converse directions do not hold in general.

(b) (see [14]) If EMFCQ holds at $\overline{x} \in M$, then $co(V(\overline{x}))$ is closed. The converse direction does not hold in general.

(c) (see [4]) Assume that EACQ (or EKTCQ or EMFCQ) holds at $\overline{x} \in M$. If $\overline{x}$ is a G-properly efficient point for MOSIP, then $\overline{x}$ is also a KT-properly efficient point for MOSIP.

(d) (see [2]) Suppose that MOSIP is a convex problem and that MOSIP fulfills SC. If $\overline{x} \in M$ is a global minimizer of (2.1) for some $\overline{\lambda} \geq 0_q$, then there exists $\overline{\mu} \in K^*$ such that

$$\overline{\lambda}^\top f(\overline{x}) = \inf_{x \in \mathbb{R}^n} \left\{ \overline{\lambda}^\top f(x) + \langle \overline{\mu}, G(x) \rangle \right\}, \ \ \langle \overline{\mu}, G(\overline{x}) \rangle = 0.$$

Optimality conditions

The first theorem establishes a necessary optimality condition of Karush-Kuhn-Tucker type for a feasible point being locally weakly efficient under various constraint qualification (statements (a) and (b)). As a generalization from finite optimization, statement (c) says that this optimality condition is also sufficient if MOSIP is assumed to be convex.

Theorem 3.1. (see [10], Theorem 4.1, Corollary 4.1, and Theorem 4.2).
(a) Let $\overline{x}$ be a locally weakly efficient point for MOSIP and assume that EMFCQ holds at $\overline{x}$. Then there exist $y^j \in Y_0(\overline{x})$, $j = 1, \ldots, s$, $s \leq n$, a vector $\lambda \geq 0_q$ and coefficients $\mu_j \geq 0$, $j = 1, \ldots, s$ such that

$$\sum_{i=1}^{q} \lambda_i Df_i(\overline{x}) + \sum_{j=1}^{s} \mu_j D_x g_j\left(\overline{x}, y^j\right) = 0. \tag{3.1}$$

(b) Let $\overline{x}$ be a locally weakly efficient point for MOSIP and assume that EKTCQ (or EACQ) holds at $\overline{x}$. If $co(V(\overline{x}))$ is closed, then there exist $y^j \in Y_0(\overline{x})$, $j = 1, \ldots, s$, $s \leq n$, $\lambda \geq 0_q$ and $\mu_j \geq 0$, $j = 1, \ldots, s$ such that (3.1) is fulfilled.

(c) Assume $\overline{x} \in M$ and that MOSIP is a convex problem. Furthermore, assume that there exist $y^j \in Y_0(\overline{x})$, $j = 1, \ldots, s$, $s \leq n$,, $\overline{\lambda} \geq 0_q$ and $\overline{\mu}_j \geq 0$, $j = 1, \ldots, s$ such that (3.1) with $\lambda = \overline{\lambda}$ and $\mu_j = \overline{\mu}_j$, $j = 1, \ldots, s$ is fulfilled. Then, $\overline{x}$ is a locally weakly efficient point for MOSIP.

In statement (a) of the previous theorem we assume EMFCQ and in statement (b) the weaker constraint qualifications EKTCQ or EACQ (see Lemma 3.1(a)). Moreover, compared to statement (a), the additional assumption in (b) is the closeness of the set $co(V(\overline{x}))$ which is not fulfilled in general for MOSIP. The next example is taken from [10] and it shows that in general EKTCQ without assuming the closeness of $co(V(\overline{x}))$ does not imply the necessary Karush-Kuhn-Tucker condition in Theorem 3.1.

Example 3.1. ([10], Example 4.1)
Let $n = 2,\ q = 2,\ m = 1,\ Y = [0, 2]$,

$$\begin{aligned} &f_1(x) = x_1 - x_2,\ \ f_2(x) = -x_2 \ \textit{and} \\ &g(x, y) = yx_1 + \sqrt{1 - (y-1)^2}x_2. \end{aligned}$$

Thus,

$$M = \left\{x \in \mathbb{R}^2 \mid x_1 \leq 0,\ \ x_2 \leq 0\right\}$$

and $\overline{x} = (0, 0)^\top$ is a locally weakly efficient solution of MOSIP. We have

$$Y_0\left(\overline{x}\right) = [0, 2],\ \ D_x g\left(\overline{x}, y\right) = \left(y, \sqrt{1 - (y-1)^2}\right)$$

and we see that the set

$$co\left(V\left(\overline{x}\right)\right) = \left\{x \in \mathbb{R}^2 \mid x_2 = 0,\ \ x_1 \geq 0\right\} \cup \left\{x \in \mathbb{R}^2 \mid\ x_1 > 0,\ \ x_2 > 0\right\}$$

is not closed. On the other hand, it is $A\left(\overline{x}, M\right) = M$, and for $d \in \mathbb{R}^2$, we get for $y \in [0, 2]$:

$$D_x g\left(\overline{x}, y\right) d \leq 0 \text{ if and only if } yd_1 + \left(\sqrt{1 - (y-1)^2}\right) d_2 \leq 0.$$

The latter property implies

$$M = \left\{d \in \mathbb{R}^2 \mid D_x g\left(\overline{x}, y\right) d \leq 0,\ \ y \in [0, 2]\right\}$$

and, hence, *EKTCQ holds at* $\overline{x}$. Now, consider the following non-negative linear combination as in (3.1):

$$\lambda_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sum_{j=1}^{s} \mu_j \begin{pmatrix} y^j \\ \left(\sqrt{1 - (y^j - 1)^2}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $y^j \in [0, 2]$ and $s \leq 2$. Any solution of the latter combination fulfills $\lambda_1 = \lambda_2 = 0$. Therefore, we have the following situation. The set $co\left(V\left(\overline{x}\right)\right)$ is not closed, EKTCQ holds at $\overline{x}$ and the Karush-Kuhn-Tucker condition (3.1) has no solution.

The next theorem refers to a necessary optimality condition of Karush-Kuhn-Tucker type for a G-properly efficient point. Note that a main difference to Theorem 3.1 is that the coefficients $\lambda_i,\ i = 1, \ldots, q$ are now strictly positive ($\lambda > 0_q$). In statement (b) we will consider separately the convex case again.

Theorem 3.2. (see [10], Theorem 4.4 and Corollary 4.4).
(a) Let $\overline{x} \in M$ be a G-properly efficient point for MOSIP and assume that EACQ holds at $\overline{x}$ and that $co(V\left(\overline{x}\right))$ is a closed set. Then, there exist $y^j \in Y_0\left(\overline{x}\right),\ j = 1, \ldots, s,\ s \leq n,$ a vector $\lambda > 0_q$ and coefficients $\mu_j \geq 0,\ j = 1, \ldots, s$ such that (3.1) is fulfilled.

(b) Assume $\overline{x} \in M$ and that MOSIP is a convex problem. Furthermore, assume that there exist $y^j \in Y_0\left(\overline{x}\right)$, $j = 1, \ldots, s,\ s \leq n,,\ \overline{\lambda} > 0_q$ and $\overline{\mu}_j \geq 0,\ j = 1, \ldots, s$ such that (3.1) with $\lambda = \overline{\lambda}$ and $\mu_j = \overline{\mu}_j,\ j = 1, \ldots, s$ is fulfilled. Then, $\overline{x}$ is a G-properly efficient point for MOSIP.

By the relationship between KT-properly efficient points and G-properly efficient points stated in Lemma 3.1(c), the assumptions (G-properly efficient point and EACQ) in statement (a) of the previous theorem imply that $\overline{x}$ is a KT-properly efficient point for MOSIP. Analogously, in statement (b) of this theorem, under the additional assumption that EACQ holds at $\overline{x}$, we would obtain that $\overline{x}$ is a KT-properly efficient point for MOSIP.

4. DUALITY FOR CONVEX PROBLEMS

Throughout this section assume that MOSIP is a convex problem. We will present two theorems concerning weak, strong and converse duality with respect to efficiency and to weak efficiency. These results are taken from the recent paper [12] that also contains results for the nonconvex case which, however, are due to their technicality beyond the scope of this tutorial note.
We start by considering the following dual multiobjective programming problem of Lagrange type [15]:

$$\text{D-MOSIP} \qquad \max\ v \ \text{ s.t. } \ (\lambda, \mu, v) \in DM$$

with the feasible set

$$DM = \left\{ (\lambda, \mu, v) \in int\ \mathbb{R}^q_+ \times K^* \times \mathbb{R}^q \mid \lambda^\top v \leq \inf_{x \in \mathbb{R}^n} \left\{ \lambda^\top f(x) + \langle \mu, G(x) \rangle \right\} \right\},$$

The following theorem states duality results for efficient points of the latter (maximize-) multiobjective programming problem as well as G-properly efficient points for MOSIP.

Theorem 4.3. (see [12], Theorem 2.3).
(a) (Weak duality). For all $x \in M$ and all $(\lambda, \mu, v) \in DM$ we have

$$\lambda^\top v \leq \lambda^\top f(x).$$

(b) (Strong duality). Assume $\overline{x} \in M$ and that MOSIP fulfills SC. If $\overline{x}$ is a G-properly efficient point for MOSIP, then there exists an efficient point $\left(\overline{\lambda}, \overline{\mu}, \overline{v}\right) \in DM$ for D-MOSIP with $f(\overline{x}) = \overline{v}$.
(c) (Converse duality). Assume that MOSIP fulfills SC and that M is compact. If $\left(\overline{\lambda}, \overline{\mu}, \overline{v}\right) \in DM$ is an efficient point for D-MOSIP, then there exists a G-properly efficient point $\overline{x} \in M$ for MOSIP with $f(\overline{x}) = \overline{v}$.

The strong duality result in statement (b) of the previous theorem holds under the assumption that $\overline{x}$ is a G-properly efficient point for MOSIP. The following example illustrates that we would not get an analogous result under the weaker assumption that $\overline{x}$ is an efficient point for MOSIP (but not a G-properly efficient one).

Example 4.2. (see [12], Example 2.4). Let MOSIP be defined as $n = 2$, $q = 2$, $Y = [0, 1]$, $x = (x_1, x_2)^\top$, $f_1(x) = x_1$, $f_2(x) = x_2$ and
$g(x, y) = x_1(y-1) - x_2\sqrt{1-(y-1)^2} - y + \sqrt{1-(y-1)^2}$.

A short calculation shows that the set of efficient points for MOSIP is

$$\left\{ x \in \mathbb{R}^2 \mid (x_1 - 1)^2 + (x_2 - 1)^2 = 1,\ 0 \leq x_i \leq 1,\ i = 1, 2 \right\}.$$

Now, we consider the efficient point $\overline{x} = (1, 0)^\top$ which is *not* a G-properly efficient point for MOSIP. This can easily be seen by the following observation (which is taken from [4, Example 2.40]). Define the efficient points

$$\begin{pmatrix} x_1(\gamma) \\ x_2(\gamma) \end{pmatrix} = \begin{pmatrix} 1 - \gamma \\ 1 - \sqrt{1 - \gamma^2} \end{pmatrix}, \ \gamma \in (0, 1].$$

Then, the term

$$\frac{f_1(\overline{x}) - f_1(x(\gamma))}{f_2(x(\gamma)) - f_2(\overline{x})} = \frac{\gamma}{1 - \sqrt{1 - \gamma^2}}$$

becomes unbounded as $\gamma \downarrow 0$. Hence, $\overline{x}$ is not a G-properly efficient point for MOSIP.
We will show now that there does not exist any efficient point $(\overline{\lambda}, \overline{\mu}, \overline{v})$ for D-MOSIP with $f(\overline{x}) = \overline{v}$. Suppose the contrary and let $(\overline{\lambda}, \overline{\mu}, f(\overline{x})) \in DM$ be an efficient point for D-MOSIP. Then $f_1(\overline{x}) = 1$, $f_2(\overline{x}) = 0$ and

$$\overline{\lambda}^\top f(\overline{x}) = \overline{\lambda}_1 \leq \inf_{x \in \mathbb{R}^2} \left\{ \overline{\lambda}^\top f(x) + \langle \overline{\mu}, G(x) \rangle \right\} \leq \inf_{x \in M} \overline{\lambda}^\top f(x).$$

By $\overline{x} \in M$, it follows that

$$\overline{\lambda}^{\top} f(\overline{x}) = \overline{\lambda}_1 = \inf_{x \in M} \overline{\lambda}^{\top} f(x) = \inf_{x \in M} \left\{ \overline{\lambda}_1 x_1 + \overline{\lambda}_2 x_2 \right\}$$

and hence

$$\overline{\lambda}_1 \leq \overline{\lambda}_1 x_1 + \overline{\lambda}_2 x_2 \text{ for all } x \in M. \tag{4.1}$$

Since $\overline{\lambda} > 0_2$, there exists always a point $\widetilde{x} \in M$ with

$$\widetilde{x} \in \left\{ x \in \mathbb{R}^2 \mid (x_1 - 1)^2 + (x_2 - 1)^2 = 1,\ 0 < x_i < 1,\ i = 1, 2 \right\}$$

and $\overline{\lambda}_1 \widetilde{x}_1 + \overline{\lambda}_2 \widetilde{x}_2 < \overline{\lambda}_1$ which contradicts (4.1).

Now, we consider weakly efficient points for the following dual multiobjective programming problem

$$\text{D}_w\text{-MOSIP} \qquad\qquad \max\ v \text{ s.t. } (\lambda, \mu, v) \in D_w M$$

with the feasible set

$$D_w M = \left\{ (\lambda, \mu, v) \in \left\{ \mathbb{R}^q_+ \setminus \{0_q\} \right\} \times K^* \times \mathbb{R}^q \mid \lambda^{\top} v \leq \inf_{x \in \mathbb{R}^n} \left\{ \lambda^{\top} f(x) + \langle \mu, G(x) \rangle \right\} \right\}.$$

The difference between the feasible sets DM and $D_w M$ is that in the latter one some (but not all) coefficients of λ may vanish. The final theorem delivers duality results for weakly efficient points for D$_w$-MOSIP as well as weakly efficient points for MOSIP.

Theorem 4.4. (see [12], Theorem 2.5).
(a) (Weak duality). There does not exist any pair of points $(x, (\lambda, \mu, v)) \in M \times D_w M$ with $f(x) < v$.
(b) (Strong duality). Assume $\overline{x} \in M$ and that MOSIP fulfills SC. If $\overline{x} \in M$ is a weakly efficient point for MOSIP, then there exists a weakly efficient point $(\overline{\lambda}, \overline{\mu}, \overline{v}) \in D_w M$ for D$_w$-MOSIP with $f(\overline{x}) = \overline{v}$.
(c) (Converse duality). Assume that MOSIP fulfills SC and that M is compact. If $(\overline{\lambda}, \overline{\mu}, \overline{v}) \in D_w M$ is a weakly efficient point for D$_w$-MOSIP, then there exists a weakly efficient point $\overline{x} \in M$ for MOSIP with $\overline{v} \geqq f(\overline{x})$.

5. CONCLUSIONS

In this tutorial note we reviewed several properties of the problem class MOSIP: solution types, constraint qualifications, optimality conditions and, under additional convexity assumptions, also weak, strong and converse duality with respect to efficiency and weak efficiency. As mentioned above, there exist only a few results towards a systematic approach to this combination of multiobjective and semi-infinite programming although there are several applications for this model. We mainly presented results from the recent papers [10, 11, 12]. For further studies note that the paper [12] also discusses duality results for the nonconvex case. In this latter case several technical assumptions (such as reduction approach and p-power transformation) are applied locally around the point under consideration. Moreover, in this case the terminology of KT- and G-*locally* properly efficient points as defined in Section 2 is used. Altogether, there are still many open questions within this class of problems which combines multiobjective and semi-infinite programming.

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