A QUEUEING SYSTEM WITH MARKOV ARRIVAL PROCESS AND EXCEPTIONAL FIRST SERVICE

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ABSTRACT

We analyse a special kind of a MAP/G/1/r FCFS queue where the first customer who arrives in the system, at the beginning of a busy period, has a service time ruled by a different distribution function than customers getting service during the busy period. An arriving customer who finds the system completely full is lost. We derive the steady state probabilities of the Markov process underlying the considered queueing system.

Key words: Queueing system, Markov Arrival Process, Supplementary variable method, Steady-state (stationary) distribution.

RESUMEN

Nosotros analizamos un tipo especial de una cola del tipo MAP/G/1/r FCFS donde el primer cliente que llega al sistema, al principio de un periodo ocupado, tiene un tiempo de servicio gobernado por una función de la distribución diferente de los clientes que son serviciados durante el periodo de ocupación. Un cliente que arriba y se encuentra el sistema completamente lleno es perdido. Nosotros derivamos las probabilidades estacionarias del proceso de Markov subyacentes en el sistema de colas considerado.

MSC: 60K25

1. INTRODUCTION

It is known that the Poisson arrival process does not match the arrival pattern in the recent communication networks. For example, the Poisson process is not appropriate for describing the packet arrival pattern because of the correlation between arriving packets in the ATM network. Therefore, with the purpose of getting a better adaptation between the models and the real problems, it is necessary that arrivals streams more complex are considered.

The Markov Arrival Process (MAP) was introduced in Neuts [1979] as a generalization of the Poisson process which is well shifted for matrix analytic and numerical investigations. Lucantoni [1991] suggested a convenient notation which is better suited for a general discussion than that which was originally used.

The MAP, a special class of tractable Markov renewal process, is a rich class of point processes that includes many well-known processes such as Poisson, PH-renewal processes, and Markov-Modulated Poisson Process (MMPP). One of the most significant features of the MAP is the underlying Markovian structure and fits ideally in the context of matrix-analytic solutions to stochastic models. Matrix-analytic methods were first introduced and studied by Neuts [1981]. As it is well known, Poisson processes are the simplest and most tractable ones used extensively in stochastic modeling. The idea of the MAP is to generalize significantly the Poisson processes and still keep the tractability for modeling purposes. Furthermore, in many practical applications, notably in communications engineering, production and manufacturing engineering, the arrivals do not usually form a renewal process. So MAP is a convenient tool to model both renewal and nonrenewal arrivals.

Also we note that MAP is useful for modeling broadband networks bursty traffic streams. There are several special cases of MAPs used in modeling data traffic, as for example the Interrupted Poisson Process (IPP) to approximate overflow traffic in finite trunks systems.

On the other hand, the queueing systems with exceptional first service may arise in many practical situations, as the server starting work after a period of idling may work slower (or faster) than the way it otherwise would. In a computer network, this may also arise because the first packet in a sequence may require special processing for route establishment, and will therefore require a different kind of service than the other packets. As related works, the reader is referred to Atencia-Moreno [2002], Koder-Mizawa [2002], Li **et al**. [1996], Welch [1964].

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There exist numerous publications considering single server queueing systems and Markovian Arrival process, for example see [2]; nevertheless, the case of exceptional first service has not been studied yet. Later on, in our paper we present an analysis of the MAP/G/1/r FCFS queue with exceptional first service. We give a procedure of finding the stationary distribution of the number of customers in the steady-state regime.

The rest of the paper is organized as follows. In Section 2 the mathematical description of the considered queueing system is given. Section 3 gives the system of differential equations using the supplementary variable method. In Section 4 the matrix solution for the queueing system is obtained. Some conclusions are presented in Section 5.

2. DESCRIPTION OF THE QUEUEING SYSTEM

Let us consider a single server queueing system at which customers arrive according to a Markov Arrival Process. The customers' service discipline is First Come First Served (FCFS). Assume the buffer capacity is r ($2 \le r < \infty$). An arriving customer who finds the system completely full is lost. The first customer to be served when a busy period starts gets a different kind of service (different distribution of service times) than the other customers served during the busy period.

The service times of customers are independent. Let $B_2(x)$ [with Laplace-Stieltjes transform $\beta_2(s)$] be the probability distribution function of the normal service time and $\beta_1(x)$ be the probability distribution function of the first customer being served in a busy period. We assume that $\beta_h(x)$ is absolutely continuous, $\beta_h(0) = 0$ and $b_h = \int_0^\infty x dB_h(x) < \infty$, h = 1, 2. Besides, we suppose that $\beta_2(-\sigma_i) \neq 0$ where $\{\sigma_i\}$ are the eigenvalues of the matrix Λ.

We shall call 1-customer to the first customer being served in each busy period and 2-customer to the second or following customers being served in each busy period. So an h-customer has a service time ruled by $B_h(x)$, h = 1, 2.

Further a symbol "·" will denote summing over all possible values of a discrete argument.

We shall briefly review MAP. The MAP is constructed by considering a two-dimensional Markov process $\{\xi(t), \nu(t): t \ge 0\}$ on the state space $\cup_{n=0}^{\infty} I_n$, where $I_n = \{(i, n): i = \overline{1,l}\}$ with an infinitesimal generator

	I_0	I_1	I_2	I_3	...
I_0	Λ	N	0	0	...
I_1	0	Λ	N	0	...
I_2	0	0	Λ	N	...
⋮	⋮	⋮	⋮	⋮	⋱

where the matrices Λ and N are $l \times l$. Λ has negative diagonal components and nonnegative off-diagonal components, and N has nonnegative components.

If $\xi(t)$ indicates the current phase of the next arrival and $\nu(t)$ represents the number of arrivals during [0,t), then the above Markov process defines a MAP where

- Λ_{ij}, $i \neq j$, is the state transition rate from state i to state j in the underlying Markov chain without an arrival.
- N_{ij} is the state transition rate from state i to state j in the underlying Markov chain with an arrival.

MAP has the following properties:

- $\Lambda^* = \Lambda + N$ is an irreducible infinitesimal generator for underlying Markov chain $\{\xi(t): t \ge 0\}$ with stationary probability vector $\vec{\delta}$ such that $\vec{\delta}^T \vec{1} = 1$ and $\vec{\delta}^T \Lambda^* = \vec{0}^T$. Obviously $\Lambda^* \vec{1} = \vec{0}$, $\Lambda^*_{ij} \ge 0$, $i \neq j$ and $\Lambda^*_{ii} < 0$.
- Λ is a stable matrix.
- The fundamental arrival rate of MAP is $\vec{\delta}^T N \vec{1} = \sum_{i=1}^{l} \delta_i N_i$.

3. SYSTEM OF DIFFERENTIAL EQUATIONS

At any arbitrary moment of time the state of the system can be characterized by the Markov process $\{\zeta(t): t \geq 0\}$ with the state space

$$\chi = \{(0,i): i = \overline{1,l} : (h,i,n,x) : h = 1, 2, i = \overline{1,l} . n = \overline{1,r+1},\ x \geq 0\}.$$

A state (0,i) of the process $\zeta(t)$ at some time moment t means that the system is idle and the Markov chain $\xi(t)$ in the state i; $\zeta(t) = (h,i,n,x)$ corresponds to the situation in which the system contains n customers, the Markov chain $\xi(t)$ is in the state i, the serving customer is an h-customer and elapsed service time is equal to x. Define $p_{o,i}(t) = P[\zeta(t) = (0,i)]$ and $P_{h,I,n}(x,t) = P[\zeta(t) \in \{h,i,n\} \otimes [0,x)\}]$.

We assume that there exist stationary probabilities $p_{o,i} = \lim_{t\to\infty} p_{0,i}(t)$ and $P_{h,i,n}(x) = \lim_{t\to\infty} P_{h,i,n}(x,t)$, and stationary probability densities $p_{h,i,n}(x) = \frac{d}{dx} P_{h,i,n}(x)$.

Note that $p_{h,i,n} = \int_0^\infty p_{h,i,n}(x)dx$ is a stationary probability that there are n customers in the system, the Markov chain $\xi(t)$ is in the state i and the serving customer is an h-customer. Let us denote $q_{h,i,n}(x,t) = [1 - B_h(x)]^{-1} p_{h,i,n}(x,t)$ for h = 1, 2, $i = \overline{1,l}$ and $n = \overline{1,r+1}$.

Now we define the vectors

$$\vec{p}_0^{\,T} = (p_{0,1}, p_{0,2}, \ldots, p_{0,l})$$

$$\vec{p}_{h,n}^{\,T} = (p_{h,1,n}, p_{h,2,n}, \ldots, p_{h,l,n});\ h = 1, 2,\ n = \overline{1,r+1}.$$

By considering the transition probabilities from t to t + Δt and using the supplementary variable method, we have the following system of differential equations:

$$\vec{0}^{\,T} = \vec{p}_0^{\,T}\Lambda + \int_0^\infty \vec{q}_{1,1}^{\,T}(x)\,dB_1(x) + \int_0^\infty \vec{q}_{2,1}^{\,T}(x)dB_2(x) \tag{1}$$

$$\frac{d}{dx}\vec{q}_{h,n}^{\,T}(x) = \vec{q}_{h,n}^{\,T}(x)\Lambda + u(n-1)\vec{q}_{h,n-1}^{\,T}(x)\,N : h = 1, 2,\ \ n = \overline{1,r} \tag{2}$$

$$\frac{d}{dx}\vec{q}_{h,r+1}^{\,T}(x) = \vec{q}_{h,r+1}^{\,T}(x)\Lambda^* + \vec{q}_{h,r}^{\,T}(x)\,N : h = 1, 2 \tag{3}$$

$$\vec{q}_{1,n}^{\,T}(0) = u(2-n)\vec{p}_0^{\,T}\,N : n = \overline{1,r+1} \tag{4}$$

$$\vec{q}_{2,n}^{\,T}(0) = \int_0^\infty \vec{q}_{1,n+1}^{\,T}(x)\,dB_1(x) + \int_0^\infty \vec{q}_{2,n+1}^{\,T}(x)\,dB_2(x) :\ n = \overline{1,r} \tag{5}$$

$$\vec{q}_{2,r+1}^{\,T}(0) = \vec{0}^{\,T} \tag{6}$$

where u(x) is a unit Heavyside function.

4. THE MATRIX SOLUTION

We introduce the sequence of functions

$$F_0(x) = e^{\Lambda x},\ F_n(x) = \int_0^x F_{n-1}(y)\,N\,e^{\Lambda(x-y)}dy,\ \ n = 1, 2, \ldots$$

Observe that $F_0(0) = I$ and $F_n(0) = 0$, $n = 1,2,...$ Note that the matrices $F_n(x)$ have a probabilistic interpretation. In fact $(F_n(x))_{ij}$ is the probability that during a time x, n customers will arrive and the input process will be in the state j, under the condition that at the initial moment it was in the state i.

Let $B_{h,n}$ be the n-th exponential moment for distribution function $B_h(x)$, h = 1,2. Observe that the matrices $B_{h,n}$ have a probabilistic interpretation. In fact $(B_{h,n})_{ij}$ is the probability that during the service of an h-customer, n customers will arrive and the input process will be in the state j, under the condition that at the initial moment it was in the state i.

For h = 1, 2 we note that

$$B_{h,0}^{*} = \int_0^\infty e^{\Lambda^* x} dB_h(x) : B_{h,n} = \int_0^\infty F_n(x) dB_h(x),\ n = 0,1,...$$

Theorem 1. The stationary probabilities satisfy

$$\vec{p}_{1,n}^{\,T}(x) = [1 - B_1(x)]\vec{p}_0^{\,T} N\, F_{n-1}(x),\ n = \overline{1, r}$$

$$\vec{p}_{1,r+1}^{\,T}(x) = [1 - B_1(x)]\vec{p}_0^{\,T} N \left\{ e^{\Lambda^* x} - \sum_{n=1}^{r} F_{n-1}(x) \right\}$$

$$\vec{p}_{2,n}^{\,T}(x) = [1 - B_2(x)] \sum_{j=0}^{n-1} \vec{q}_{2,n-j}^{\,T}(0) F_j(x),\ \ n = \overline{1,r}$$

$$\vec{p}_{2,r+1}^{\,T}(x) = [1 - B_2(x)] \sum_{n=1}^{r} \left[\vec{q}_{2,n}^{\,T}(0) e^{\Lambda^* x} - \vec{q}_{2,n}^{\,T}(x) \right]$$

where $\vec{q}_{2,n}^{\,T}(0) = \vec{p}_0^{\,T}\ Q_n,\ n = \overline{1,r}$ and

$$Q_1 = -(N\, B_{1,0} + \Lambda) B_{2,0}^{-1} \tag{7}$$

$$Q_n = \left(Q_{n-1} - \sum_{j=1}^{n-1} Q_{n-j} B_{2,j} - N B_{1,n-1} \right) B_{2,0}^{-1},\ n = \overline{2,r}\ . \tag{8}$$

The system of equations $\vec{p}_0^{\,T} \left\{ \left(\sum_{n=1}^{r} Q_n \right) (I - B_{2,0}^{*}) - N\, B_{1,0}^{*} - \Lambda \right\} = \vec{0}^{\,T}$ together with the expression $1 - p_0 = b_1 q_1(0) + b_2 q_2(0)$ determines the vector $\vec{p}_0$ exactly.

Proof: We can easily show that for fixed h (h = 1,2) the solution of equation (2) is $\vec{q}_{h,n}^{\,T}(x) = \sum_{j=0}^{n-1} \vec{q}_{h,n-j}^{\,T}(0) F_j(x),\ \ n = \overline{1,r}$ and using (4) we get

$$\vec{q}_{1,n}^{\,T}(x) = \ \vec{p}_0^{\,T} N\, F_{n-1}(x),\ n = \overline{1,r} \tag{9}$$

$$\vec{q}_{2,n}^{\,T}(x) = \sum_{j=0}^{n-1} \vec{q}_{2,n-j}^{\,T}(0) F_j(x),\ \ n = \overline{1,r}\ . \tag{10}$$

Summing equations (2) – (3) over $n = \overline{1, r+1}$ and integrating the result, we have

$$\sum_{n=1}^{r+1} \vec{q}^{T}_{h,n}(x) = \left(\sum_{n=1}^{r+1} \vec{q}^{T}_{h,n}(0) \right) e^{\Lambda^* x}, \; h = 1,2 \tag{11}$$

$$\vec{q}^{T}_{h,r+1}(x) = \vec{q}^{T}_{h,r+1}(0)\, e^{\Lambda^* x} + \sum_{n=1}^{r} \left[\vec{q}^{T}_{h,n}(0)\, e^{\Lambda^* x} - \vec{q}^{T}_{h,n}(x) \right], \; h = 1,2$$

but $\vec{q}^{T}_{h,r+1}(0) = \vec{0}^{T}$, h = 1,2 then

$$\vec{q}^{T}_{h,r+1}(x) = \sum_{n=1}^{r} [\vec{q}^{T}_{h,n}(0) e^{\Lambda^* x} - \vec{q}^{T}_{h,n}(x)], \; h = 1,2 \, .$$

Besides, if we take into account (4) and (9), we obtain

$$\vec{q}^{T}_{1,r+1}(x) = \vec{p}^{T}_{0} N e^{\Lambda^* x} - \sum_{n=1}^{r} \vec{p}^{T}_{0} N F_{n-1}(x) = \vec{p}^{T}_{0} N \left\{ e^{\Lambda^* x} - \sum_{n=1}^{r} F_{n-1}(x) \right\} .$$

We now focus on determining $\vec{q}^{T}_{2,n}(0)$ for $n = \overline{1,r}$. By substituting (9) - (10) into (1) and (5), we get

$$\vec{0}^{T} = \vec{p}^{T}_{0} \Lambda + \vec{p}^{T}_{0} N B_{1,0} + \vec{q}^{T}_{2,1}(0) B_{2,0}$$

$$\vec{q}^{T}_{2,n}(0) = \vec{p}^{T}_{0} N B_{1,n} + \sum_{j=0}^{n} \vec{q}^{T}_{2n+1-j}(0) B_{2,j}, \; n = \overline{1,r-1} \, .$$

Therefore, we have the following system of equations for $\vec{q}^{T}_{2,n}(0)$, n = $\overline{1,r}$:

$$-\vec{p}^{T}_{0} N B_{1,0} - \vec{p}^{T}_{0} \Lambda = \vec{q}^{T}_{2,1}(0) B_{2,0}$$

$$-\vec{p}^{T}_{0} N B_{1,1} = \vec{q}^{T}_{2,1}(0) B_{2,1} - I) + \vec{q}^{T}_{2,2}(0) B_{2,0}$$

$$-\vec{p}^{T}_{0} N B_{1,n-1} = \sum_{j=1}^{n-2} \vec{q}^{T}_{2,j}(0) B_{2,n-j} + \vec{q}^{T}_{2,n-1}(0)(B_{2,1} - I) + \vec{q}^{T}_{2,n}(0) B_{2,0}, \; n = \overline{3,r}$$

which can be written in the matrix form

$$(-\vec{p}^{T}_{0} N B_{1,0} - \vec{p}^{T}_{0} \Lambda, -\vec{p}^{T}_{0} N B_{1,1}, \ldots, -\vec{p}^{T}_{0} N B_{1,r-1}) = \left(\vec{q}^{T}_{2,1}(0), \vec{q}^{T}_{2,2}(0), \ldots, \vec{q}^{T}_{2,r}(0) \right) \begin{pmatrix} B_{2,0} & B_{2,1} - I & \cdots & B_{2,r-1} \\ 0 & B_{2,0} & \cdots & B_{2,r-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{2,0} \end{pmatrix} .$$

Since the matrix of coefficients for this system has a triangular structure in blocks, if the matrix $B_{2,0}$ is inversible, then the unknown vectors $\vec{q}_{2,n}(0)$, $n = \overline{1,r}$ can be expressed in terms of the vector $\vec{p}_0$.

It is known that if σ_i is an eigenvalue of the matrix Λ, then $\beta_2(-\sigma_i)$ is an eigenvalue of the matrix $\beta_2(-\Lambda) = B_{2,0}$. And by hypothesis $\beta_2(-\sigma_i) \neq 0$, $i = \overline{1,l}$, so the matrix $B_{2,0}$ is inversible and there exists $B^{-1}_{2,0}$.

Now we are going to look for the solution for this system of equations in the form

$$\vec{q}^{T}_{2,n}(0) = \vec{p}^{T}_{0} Q_n, \; n = \overline{1,r} \tag{12}$$

where the matrices Q_n are determined by the next formulae:

for n = 1, $\vec{q}_{2,1}^T(0) = -\vec{p}_0^T(N B_{1,0} + \Lambda)B_{2,0}^{-1}$;

for n = 2, $\vec{p}_0^T N B_{1,1} = \vec{p}_0^T Q_1(B_{2,1} - I) + \vec{q}_{2,2}^T(0)B_{2,0}$;

$$\vec{q}_{2,2}^T(0) = \vec{p}_0^T(Q_1 - Q_1 B_{2,1} - NB_{1,1})B_{2,0}^{-1}$$

for n = $\overline{3,r}$ $\vec{p}_0^T NB_{1,n-1} = \sum_{j=1}^{n-2}\vec{p}_0^T Q_j B_{2,n-j} + \vec{p}_0^T Q_{n-1})B_{2,1} - I) + q_{2,n}^T(0)B_{2,0}$

$$\vec{q}_{2,n}^T(0) = \vec{p}_0^T\left(Q_{n-1} - \sum_{j=1}^{n-1} Q_{n-j}B_{2,j} - N B_{1,n-1}\right)B_{2,0}^{-1}$$

and so we have the expressions (7) - (8) for Q_n, $n = \overline{1,r}$.

Finally, it will be enough to determine the vector $\vec{p}_0$. Summing equation (5) over $n = \overline{1,r}$ and adding the result to equation (1), we have

$$\sum_{n=1}^{r}\vec{q}_{2,n}^T(0) = \int_0^\infty \sum_{n=1}^{r+1}\vec{q}_{1,n}^T(x)dB_1(x) + \int_0^\infty \sum_{n=1}^{r+1}\vec{q}_{2,n}^T(x)dB_2(x) + \vec{p}_0^T\Lambda.$$

Now using (4), (6) and (11) - (12) leads to

$$\vec{p}_0^T\left\{\left(\sum_{n=1}^{r} Q_n\right)(I - B_{2,0}^*) - N B_{1,0}^* - \Lambda\right\} = \vec{0}^T. \tag{13}$$

The vector $\vec{p}_0$ is found from {13) with accuracy up to a constant, for whose determination we are going to use an equivalent relation to the normalization condition. By multiplying equation (11) on the right by the vector 1 we obtain

$$\sum_{n=1}^{r+1}\vec{q}_{h,n}^T(x)\vec{1} = \left(\sum_{n=1}^{r+1}\vec{q}_{h,n}^T(0)\right) e^{\Lambda^* x}\vec{1},\ h = 1,2$$

but $e^{\Lambda^* x}\vec{1} = \left(\sum_{n=0}^{\infty}\frac{\Lambda^{*n}}{n!}\right)\vec{1} = (I + \Lambda^* + ... + \frac{\Lambda^{*n}}{n!} + ...)\vec{1} = \vec{1}$ and thus $q_h(x) = q_h(0),\ h = 1,2$

$$\int_0^\infty [1 - B_h(x)]q_h(x)dx = \int_0^\infty [1 - B_h(x)]q_h(0)dx,\ h = 1,2$$

$$p_h = b_h q_h(0),\ h = 1,2$$

$$1 - p_0 = b_1 q_1(0) + b_2 q_2(0). \tag{14}$$

Therefore, if the equation (12) is used to determine $\vec{q}_{2,n}(0), n = \overline{1,r}$, the system of equations (13) together with the relation (14) determines the vector $\vec{p}_0$ exactly. □

Corollary 1. The main performance measures of the system are the following:

1. The server/system is idle with probability p_0.

2. The server/system is busy with probability 1 - p_0.

3. The probability that the server is busy by a customer with exceptional service is

$$p_1 = b_1 \vec{p}_0^T N \ldots \vec{1} \tag{15}$$

4. The probability that the server is busy by a customer with normal service is

$$p_2 = b_2 \vec{p}_0^T \left(\sum_{n=1}^{r} Q_n \right) \vec{1}. \tag{16}$$

Proof.

Multiplying equation (11) on the right by the vector $\vec{1}$ we have

$$\sum_{n=1}^{r+1} \vec{q}_{h,n}^T(x)\vec{1} = \left(\sum_{n=1}^{r+1} \vec{q}_{h,n}^T(0) \right) e^{\Lambda^* x}\vec{1},\ h = 1,2$$

$$q_h(x) = \left(\sum_{n=1}^{r+1} \vec{q}_{h,n}^T(0) \right) \vec{1},\ h = 1,2$$

$$\int_0^\infty [1 - B_h(x)] q_h(x) dx = \left(\sum_{n=1}^{r+1} \vec{q}_{h,n}^T(0) \right) \vec{1} \int_0^\infty [1 - B_h(x)]\, dx,\ h = 1,2.$$

$$p_h = b_h \left(\sum_{n=1}^{r+1} \vec{q}_{h,n}^T(0) \right) \vec{1},\ h = 1,2.$$

Now using the equations (4), (6) and (12) we obtain (15) - (16). □

Corollary 2. The procedure of finding the stationary distribution of the number of customers in the steady-state regime is the following:

$$p_0 = 1 - b_1 \vec{p}_0^T N \vec{1} - b_2 \vec{p}_0^T \left(\sum_{n=1}^{r} Q_n \right) \vec{1}$$

$$p_1 = -\vec{p}_0^T (\Lambda^* + Q_1) \Lambda^{-1} \vec{1}$$

$$p_n = \vec{p}_0^T \{ (-1)^n (N + Q_1)(I + \Lambda^{-1}N)(\Lambda^{-1}N)^{n-2} - Q_n \} + \Lambda^{-1}\vec{1} +$$

$$+ u(n-2) \vec{p}_0^T \left\{ \sum_{j=2}^{n-1} Q_j)(I + \Lambda^{-1}N)(-\Lambda^{-1}N)^{n-j-1} \right\} \Lambda^{-1}\vec{1},\ n = \overline{2,r}$$

$$p_{r+1} = 1 - p_0 - \sum_{n=1}^{r} p_n .$$

Proof: For h = 1, 2, n = $\overline{1,\ldots,r}$, multiplying (2) by 1-$B_h(x)$ and integrating it over x we obtain

$$\int_0^\infty [1-B_h(x)]\frac{d}{dx}\vec{q}^T_{h,n}(x)dx = \int_0^\infty \vec{p}^T_{h,n}(x)dx\,\Lambda + u(n-1)\int_0^\infty \vec{p}^T_{h,n-1}(x)\,dx\,N$$

but

$$\int_0^\infty [1-B_h(x)]\frac{d}{dx}\vec{q}^T_{h,n}(x)dx = -\vec{q}^T_{h,n}(0) + \int_0^\infty \vec{q}^T_{h,n}(x)dB_h(x)$$

and so

$$-\vec{q}^T_{h,n}(0) + \int_0^\infty \vec{q}^T_{h,n}(x)dB_h(x) = \vec{p}^T_{h,n}\Lambda + u(n-1)\vec{p}^T_{h,n-1}N.$$

Next, summing the above equality over h = 1, 2 for n = 1 and adding the result to (1), we have

$(\vec{p}^T_{1,1} + \vec{p}^T_{2,1})\Lambda = -(\vec{p}^T_0\Lambda^* + \vec{q}^T_{2,1}(0))$ for $n = \overline{2,r}$ and adding the result to $\vec{q}^T_{2,n-1}(0) = \int_0^\infty \vec{q}^T_{1,n}(x)dB_1(x) + \int_0^\infty \vec{q}^T_{2,n}(x)dB_2(x)$, we have $-\vec{q}^T_{2,n}(0) + \vec{q}^T_{2,n-1}(0) = (\vec{p}^T_{1,n} + \vec{p}^T_{2,n})\Lambda + (\vec{p}^T_{1,n-1} + \vec{p}^T_{2,n-1})N,\ n = \overline{2,r}$

Now using (12) we can write

$$\vec{p}^T_{1,1} + \vec{p}^T_{2,1} = \vec{p}^T_0(\Lambda^* + Q_1)\Lambda^{-1}$$

$$\vec{p}^T_{1,n} + \vec{p}^T_{2,n} = -\left\{\vec{p}^T_0(Q_n - Q_{n-1}) + (\vec{p}^T_{1,n-1} + \vec{p}^T_{2,n-1})N\right\}\Lambda^{-1},\ n = \overline{2,r}.$$

Proceeding by recurrence we get

$$\vec{p}^T_{1,1} + \vec{p}^T_{2,1} = -\vec{p}^T_0(\Lambda^* + Q_1)\Lambda^{-1}$$

$$\vec{p}^T_{1,n} + \vec{p}^T_{2,n} = \vec{p}^T_0\{(-1)^n(N + Q_1)(I + \Lambda^{-1}N)(\Lambda^{-1}N)^{n-2} - Q_n\}\Lambda^{-1} +$$

$$+ u(n-2)\vec{p}^T_0\left\{\sum_{j=2}^{n-1} Q_j(I + \Lambda^{-1}N)(-\Lambda^{-1}N)^{n-j-1}\right\}\Lambda^{-1},\ n = \overline{2,r}.$$

which completes the proof of corollary 2. □

Remark 1. Assume $B_h(x)$, h = 1, 2 are of phase type with irreducible PH-representation $(\vec{\beta}_h, M_h)$ where $\vec{\beta}_h$ is an $m_h \times 1$ vector, $\vec{\beta}^T_h\,\vec{1} = 1$ and M_h is an $m_h \times m_h$ matrix. Denote $\vec{\mu}_h = -M_h\vec{1}$. Then for h = 1, 2 we have

$$B_{h,n} = (-1)^{n+1}(I \otimes \vec{\beta}^T_h)[(\Lambda \oplus M_h)^{-1}(N \otimes I)]^n(\Lambda \oplus M_h)^{-1}(I \otimes \vec{\mu}_h),\ n = 0,1,...$$

$$B^*_{h,0} = -(I \otimes \vec{\beta}^T_h)(\Lambda^* \oplus M_h)^{-1}(I \otimes \vec{\mu}_h).$$

The symbol ⊗ denotes the Kronecker product of two matrices, i.e., A ⊗ B denotes the matrix made up of blocks $a_{ij}B$. The symbol ⊕ denotes the Kronecker sum of two matrices: A ⊕ B = A ⊗ I + I ⊗ B.

5. CONCLUSIONS AND FUTURE WORK

The importance of this paper is in the direct resolution of the system of differential equations. From the solution we can obtain, among others things, all the moments for the stationary distribution of the number of customers in the system. Besides, the procedure of finding the solution can be easily implemented in several programming languages and, on this way, we can make comparissons between systems with exceptional distribution functions sensitively different.

Moreover the previous formulae can be used for the approximate calculus (with accuracy up to a known ε of the stationary features of the respective queueing system with infinite buffer capacity (for values of r large enough).

Finally, we observe that a Batch Markovian Arrival Process (BMAP) is a MAP where the arrivals can occur in random-sized batches. As future work, we would like to analyse the BMAP/G/1/r FCFS queue with exceptional first service.

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