A QUEUEING SYSTEM UNDER LCFS PR DISCIPLINE WITH MARKOVIAN ARRIVAL PROCESS AND GENERAL TIMES OF SEARCHING FOR SERVICE

I. Atencia, Department of Applied Mathematics, E. T. S. I. Informatics, University of Malaga, Spain
N .H. Phong, Department of Probability Theory and Mathematical Statistics, Peoples' Friendship University of Russia, Moscow, Russia

ABSTRACT
We consider a single server queueing system with repeated attempts in which customers arrive according a Markov Arrival Process (MAP) and with a LCFS PR discipline. The service times are independent and have a common general distribution. After service completion time the server initiates his search time with an arbitrary distribution function. We consider two cases where the maximum number of repeated customers waiting in the orbit to seek service again is limited by $r(r < \infty)$ or can be unlimited $(r = \infty)$. We derive the steady state probabilities of the embedded Markov chain at service completion times of the process and also the steady state probabilities of the underlying Markov linear process.

Key words: Queueing system, Markov arrival process, customer-searching server, general time of searching.

RESUMEN
Consideramos un servicio de cola con un solo servidor con intentos repetidos en el cual los usuarios arriban de acuerdo a un Proceso de Arribos de Markov (PAM) y con disciplina LCFS PR. Los tiempos de servicio son independientes y poseen la misma distribución general. Después del terminar el servicio el servidor inicia su tiempo de búsqueda con una distribución arbitraria. Consideramos dos casas donde el número máximo de usuarios repetidos que esperan en la órbita buscando servicio nuevamente esta limitado por r $(r < \infty)$. Nosotros derivamos las probabilidades de estado estable de la cadena de Markov inmersa en los tiempos de completamiento del proceso y también las probabilidades de estabilidad del estado del proceso linear de Markov subyacente.

MSC: 60K25.

1. INTRODUCTION

We consider an single-server queueing system without buffer. Customers flow entering the system forms a Markov Arrival Process (MAP), defined by the matrices Λ and N of size 1. The entry $\Lambda_{i,j}, i \neq j$ of matrix Λ is the transition intensity of the generation process from the phase i to the phase j with no new arrivals, while the entry $N_{i,j}$ of the matrix N is the transition intensity from the phase i to the phase j with new arrival. Additionally, we introduce the matrices $\Lambda^* = \Lambda + N$ and suppose that the matrix Λ^* is a conservative matrix of the transition rates of the controlled Markov process $\{\xi(t), t \geq 0\}$ in the space of the generation phases $\{1, 2,...,l\}$, and N is not null matrix.

The service times of customers are independent random variables with a common arbitrary distribution function (DF) $B_1(x)$. When the server becomes free, that is, when a service is completed, the server starts his search time even if the orbit is empty, with an arbitrary distribution function $B_0(x)$. An arriving customer who finds the server idle obtains service immediately or expels the customer in the service, otherwise, he begins his service. The expelled customer joins a group of unsatisfied customers, that we call orbit, in order to get service later. At a search completion moment the server starts new service if there are customers' on the orbit or makes new search if the orbit is empty. We assume that $B_s(0) = 0$, s = 0, 1 and $\int_0^\infty t dB_s(t) = 1/\mu_s < \infty$.

A customer that arrives when the server is busy and the orbit is occupied fully removes the customer standing in service from the server and starts his own service.

Neuts and Ramalhoto were the first who introduced the queueing model with customer-searching server. In [Neuts **et al**. 1984] they considered the system M/G/1 with infinite orbit. The system $M_2/G_2/1/r$ with a finite buffer and priority search for customers is analyzed in [Neuts, 1981], where recurrent formulas for stationary state probabilities of the system are derived. In [7-8] the systems $M/HG_K/1/r/s$ and $M_K/ HG_K/1/r/s$ with one-dimentional or pluri-dimentional Poisson flow, finite buffer and orbit from which retrial customers make attempts to rejoin to the queue in buffer. The server seeks customers for service from the queue in buffer.

In [Atencia, 2001] the system M/G/1 with LCFS PR discipline of service and general searching times is studied. In this paper we extended the obtained results in [Atencia, 2001] to the case where the flow of customers is MAP.

2. THE CASE OF FINITE ORBIT

In this section we consider the case where the maximum number of customers in the orbit are limited by a number r, $(r < \infty)$.

The Embedded Markov chain

Firstly we introduce auxiliary matrices B_s, K_s, s = 0,1 that will be needed for further consideration

$$B_s = \int_0^\infty e^{\Lambda x} dB_s(x),$$

$$K_s = \int_0^\infty e^{\Lambda x} N(1 - B_s(x))dx = -(I - B_s)\,\Lambda^{-1}N.$$

The matrix entry $(K_s)_{i,j}$ is the probability of successful completion of searching (s = 0) or serving (s = 1) process given that the generation passed to the phase j at the completion moment and started in the phase i at the beginning.

The matrix entry $(K_s)_{i,j}$ is the probability of interruption of searching (s = 0) or serving (s = 1) process by a new arrival customer given that the generation passed to phase j at the interruption moment and started in the phase i at the beginning.

Now we consider the embedded Markov chain $\{\gamma_k, k \geq 0\}$ induced by the moments immediately after changes of server states (interruption or completion of searching or serving processes).

The Markov chain $\{\gamma_k, k \geq 0\}$ has the following set of states

$$y = \{(i,s,n),\ i = \overline{1,l}\ ,\ s = 0,1,\ n = \overline{0,r}\ \}.$$

The state (i, s, n) means that at time t there are n customers in the orbit, the generation process is in the phase i, the server operates on the searching (s = 0) or serving (s = 1) regime.

We denote by $\pi_{i,s,n}$ the stationary probabilities of state (i, s, n) and introduce their row derivative vectors

$$\vec{\pi}_{s,n} = (\pi_{1,s,n},\cdots,\pi_{1,s,n})\ ,\ \vec{\pi}_n = (\vec{\pi}_{0,n},\vec{\pi}_{1,n}).$$

Let

$$Q_{(i,s,n)(i',s',n')} = \mathrm{Prob}(\gamma_k = (i', s', n')|\ \gamma_{k-1} = (i, s, n))$$

We define the probability transition matrix $Q = (Q_{k,n})_{k,n=\overline{0,r}}$ for the embedded Markov chain $\{\gamma_k, k \geq 0\}$:

$$Q_{0,0} = \begin{pmatrix} B_0 & K_0 \\ B_1 & 0 \end{pmatrix},\ Q_{0,1} = \begin{pmatrix} 0 & 0 \\ 0 & K_1 \end{pmatrix},\ Q_{n,n\text{-}1} = \begin{pmatrix} 0 & B_0 \\ 0 & 0 \end{pmatrix},$$

$$Q_{n,n} = \begin{pmatrix} 0 & K_0 \\ B_1 & 0 \end{pmatrix},\ Q_{n,n+1} = \begin{pmatrix} 0 & 0 \\ 0 & K_1 \end{pmatrix},\ n = \overline{1, r-1},$$

$$Q_{r,r\text{-}1} = \begin{pmatrix} 0 & B_0 \\ 0 & 0 \end{pmatrix},\ Q_{r,r} = \begin{pmatrix} 0 & K_0 \\ B_1 & K_1 \end{pmatrix}.$$

where 0 is a matrix of zeros. Indexes k, n, show the number of customers in the orbit.

All the other block-matrix elements of the matrix Q are zeros. Therefore the matrix Q has the form

$$Q = \begin{pmatrix} Q_{0,0} & Q_{0,1} & 0 & \dots & 0 & 0 \\ Q_{1,0} & Q_{1,1} & Q_{1,2} & \dots & 0 & 0 \\ 0 & Q_{2,1} & Q_{2,2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & Q_{r,r-1} & Q_{r,r} \end{pmatrix}.$$

Now we denote by

$$\vec{\pi} = (\vec{\pi}_0, \dots, \vec{\pi}_r)$$

the vectors of stationary probability of Markov chain. The system of equilibrium equations (SEE) can be written in the form

$$\vec{\pi} Q = \vec{\pi}$$

with the normalization condition

$$\vec{\pi}\vec{1} = 1.$$

For solving SEE we introduce the n-system, which is obtained from the original system by observing its behaviour on the states when the number of customers in the orbit don't exceed n.

Consideration of relation between the n-systems and the original system help us to solve SEE by the following way. First we transform matrix Q into matrix Q'. In each step m, where m runs from r to 1, we transform the principle minor of size m by the following changes:

matrix $Q_{m,m}$ by

$$Q'_{m,m} = (I - Q_{m,m})^{-1},$$

matrix $Q_{m,m\text{-}1}$ by

$$Q'_{m,m-1} = Q'_{m,m} Q_{m,m-1},$$

matrices $Q_{k,m-1}$, k = m – 2, m – 1, by

$$Q'_{k,m-1} = Q_{k,m-1} + Q_{k,m} Q'_{m,m-1}.$$

All the other matrices $Q_{k,m}$ are not changed. For compact exposition we will not introduce new notations and we keep the old name for the obtained after each step matrix Q' as Q. The proof of invertibility of the matrix $I - Q_{m,m}$ is straightforward. The process continue up to the minor $Q_{0,0}$.

After this, we solve the SEE to within a constant factor:

$$\bar{\pi}_0 = \vec{\pi}_0 Q_{0,0}.$$

Then, we use the recurrent formulas to establish the vectors $\bar{\pi}_n$, $n = \overline{1,r}$ to within a constant factor:

$$\bar{\pi}_n = \vec{\pi}_{n-1} Q_{n-1,n} Q_{n,n}.$$

Finally we define the value of the constant using the normalization condition.

Linear Markov process and its steady-state distribution

The behaviour of the investigated queueing system can be described by a linear Markov process $\{\gamma(t), t \geq 0\}$ of the state set

$$X = \{(i,s,n,x), i = \overline{1,l}, \ s = 1,2, n = \overline{0,r}, \ x \geq 0\}.$$

The states of process $\{\gamma(t), t \geq 0\}$ keep the same meaning as before except x-the parameter showing the elapsed time from the begining of search or service.

We denote by $p_{i,0}$-stationary probability of state (i,0); by $p_{i,s,n}(x)$ - stationary probability density of state (i,s,n,x); and by $p_{i,s,n} = \int_0^\infty p_{i,s,n}(x)dx$ - stationary probability of state (i,s,n,x) with no regards on the elapsed time x.

We introduce also the row vectors of stationary probabilities

$$\vec{p}_{s,n}(x) = (P_{1,s,n}(x),\ldots,P_{l,s,n}(x)),$$

$$\vec{p}_{s,n} = (P_{1,s,n},\ldots,p_{l,s,n}).$$

Knowing the stationary probabilities of states of embedded Markov chain and using the methods of the renewal theory we can define the stationary distribution for the linear Markov process $\{\gamma(t), t \geq 0\}$.

For this purpose we consider the mean time $\bar{t}$ of the interval between two adjacent changed moments of the embedded Markov chain. In the stationary system mode $\bar{t}$ can be defined by the formulas

$$\bar{t} = \sum_{n=0}^{r}\sum_{s=0}^{l} \vec{\pi}_{s,n}(-\Lambda^{-1} + b_s I)\vec{1},$$

where $b_s = \int_0^\infty t dB_s(t)$ - the mean time of successful searching (s = 0) or serving (s = 1) processes.

Therefore the stationary probability density $\vec{p}_{s,n}(x)$, s = 0,1, can be written in the term of $\vec{\pi}_{s,n}$ - stationary probabilities of embedded Markov chain:

$$\vec{p}_{s,n}(x) = \frac{1}{\bar{t}} \vec{\pi}_{s,n} e^{\Lambda x}(1 - B_s(x)).$$

Finally we get the stationary probabilities of system states without regards on the elapsed time

$$\vec{p}^{T}_{s,n} = \frac{1}{\bar{t}} \vec{\pi}^{T}_{s,n} \int_0^\infty e^{\Lambda x}(1 - B_s(x))dx$$

$$= -\frac{1}{\bar{t}} \vec{\pi}^{T}_{s,n} (I - B_s)\Lambda^{-1}.$$

3. THE CASE OF INFINITE ORBIT

In this section we consider the same queueing system as in the 1st section, but now the length of orbit is infinite. In this case the probability transition matrix Q of embedded Markov chain $\{\gamma_k,\ k \geq 0\}$ has the following form

$$Q = \begin{pmatrix} Q_{0,0} & Q_{0,1} & 0 & \cdots \\ Q_{1,0} & Q_{1,1} & Q_{1,2} & \cdots \\ 0 & Q_{2,1} & Q_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We denote by n-level the subset of states of embedded Markov chain, where the number of orbiting customers in the orbit is n, and consider the matrix $R_{n,n-1}$, the entries of which are the probabilities of the first return to n -1 level from a state of n-level; at the same time the phases changed in correspondence with indexes.

Having

$$R_{n,n-1} = Q_{n,n-1} + Q_{n,n}R_{n,n-1} + Q_{n,n+1}R_{n+1,n}R_{n,n-1},\ n \geq 1.$$

we notice that all the matrices $Q_{n,n-1}$, $Q_{n,n,}$ $Q_{n,n+1}$ are the same for all n, $n \geq 1$, and the index n of matrix Q is indefinite. Therefore all the matrices $R_{n,n-1}$ are equal to the matrix which we denote by R.

We introduce also the following notations

$$A = Q_{n,n-1},$$

$$B = -I + Q_{n,n},$$

$$C = Q_{n,n+1}.$$

and rewrite the equation for R in the term

$$0 = A + BR + CR^2.$$

where matrix D = A + B + C - transition matrix of an Markov process.

Suppose that the matrix D is irreducible. The solution of this well known equation can be found in different works, for example [2-4]. Suppose that the matrix R is known. We consider the r-system, which is obtained from the original system by observing it's behaviour on the states when the number of customers on the orbit don't exceed r. The r-system differs from the original system by limited lenghth r of orbit. The transition matrix Q_r for embedded Markov chain $\{\gamma_k^{(r)},\ k \geq 0\}$ differs from the transition matrix Q of the embedded Markov chain, described in the 1st section, in matrix element $Q_{r,r}$ only.

$$Q_{r,r} = Q_{r,r} + Q_{r,r+1}R$$

Therefore we have the similar r-system as in the section 1. The stationary probabilities of this system differ from the stationary probabilities of the original system in a constant only.

Further analysis of the system does not differ from the analysis in the section 1 and is not repeated.

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