ON WEAKLY EFFICIENT BOUNDS FOR MULTIPLE OBJECTIVE PROGRAMMING PROBLEMS

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ABSTRACT
In this paper we analyze the consequences produced by introducing the notions of weakly efficient bounds and suprema in the multiple objective programming model. These concepts can be seen as generalizations of their scalar counterparts and some properties and results concerning them are obtained. Through the developed theory it is shown that under certain assumptions, we can get a polarity relation, in a weakly efficient sense, between the multiobjective convex programming problem and the one that arises in computing its weakly efficient suprema. This provides us with a restricted dual weakly vector problem definition for the linear case. Some apparently new theorems of the alternative given in this work have special relevance in this issue.

Key words: Multiple objective programming, weakly efficient solution, weakly efficient bound, weakly efficient supremum, theorems of the alternative, duality in vector optimization.

MSC: 90C29, 90C99.

RESUMEN
En este artículo analizamos las consecuencias que se derivan al introducir las nociones de cotas y supremos débilmente eficientes en el modelo de programación multiobjetivo. Estos conceptos pueden considerarse como las generalizaciones de sus análogos escalares, habiéndonos sido posible obtener algunas propiedades y resultados relativos al caso vectorial. La teoría desarrollada muestra que bajo ciertas hipótesis podemos obtener una relación de polaridad, en un sentido débilmente eficiente, entre el problema de programación multiobjetivo convexo y el que surge al calcular todos sus supremos débilmente eficientes. Esto nos proporciona una definición restringida de problema vectorial dual en sentido débil para el caso lineal. Ciertos teoremas de la alternativa aparentemente inéditos presentados en este trabajo resultan fundamentales para el desarrollo de la teoría aquí expuesta.

1. INTRODUCTION

The multiple objective programming problem (MOP) involves the simultaneous maximization of $k \geq 2$ noncomparable criterion functions, $z: R^n \to R^k$, over a feasible region, $X \subseteq R^n$, which is given in an implicit way. Thus, this problem may be written as:

$$\max \{z(x) / x \in X\} \tag{1}$$

The set $z(X) = \{z(x) / x \in X\}$ is called *outcome set* or *criterion set*.

Before going further, for convenience, let us introduce the following notation. Let $x, y \in R^n$, then:

1. $x \leq y \iff x_j \leqq y_j, \forall j \in \{1,\ldots,n\}$.
2. $x \leq y \iff x \leqq y,\ x \neq y$.
3. $x < y \iff x_j < y_j, \forall j \in \{1,\ldots,n\}$
4. $R^n_+ \iff \{x \in R^n / x \geqq 0\}$.
5. $R^n_{++} \iff \{x \in R^n / x > 0\}$.
6. $R^n_- \iff \{x \in R^n / x \leqq 0\}$.

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7. $R^n_{--} \Leftrightarrow \{x \in R^n / x < 0\}$.

8. $e \Leftrightarrow$ Vector whose components are each equal to 1.

When all the objective functions that appear in (1) are linear and X is a polyhedron, we have a multiple objective linear programming problem (MOLP). Without loss of generality, we can assume that the formulation of a MOLP is as follows:

$$\max\{Cx / x \in X\} \tag{2}$$

where $X = \{x \in R^n_+ / Ax = b\}$, being $A \in R^{m\times n}$, $b \in R^{m\times 1}$ and $C \in R^{k\times n}$, fixed real matrices.

Another important version of problem (1) arises when X is a convex closed set and z is concave. Then we have the so-called convex multiobjective programming problem (CMP).

There are many different approaches for analyzing and solving multiple objective programs, such as vector maximization, interactive programming or goal programming among others (the reader is referred to, e.g., Evans (1984), Steuer (1986), and the references therein). In almost all of them, the concept of nondominated or weakly nondominated solution has played a prominent role. Particularly, the analysis we are about to present will be concerned on the second of these concepts, which is formally defined as follows:

Definition 1.1 *We say that* $\bar{y} \in Y \subseteq R^k$ *is a weakly nondominated point of* Y *if, and only if, there exists no* $y \in Y$ *such that* $y > \bar{y}$*. Otherwise it is a weakly dominated point of* Y*.*

Let P be a MOP. We will refer to the weakly nondominated points of the outcome set z(X) as weakly nondominated solutions of P. The above concept can be translated to the decision set X. Thus, we can get:

Definition 1.2 *A point* $\bar{x} \in X$ *is said to be a weakly efficient solution of* P *if, and only if,* $z(\bar{x})$ *is a weakly nondominated solution of* P*.*

Let WE^P denote the set of all weakly efficient solutions for problem P.

In this research we introduce a new theoretical frame that has been developed by Jorge (2002b) as a part of his Ph.D. Thesis, with the aim of achieving a dual formulation (in a weakly efficient sense) of a vector program. This theory is based on the concepts of weakly efficient bounds and suprema for the MOP, which have been defined in a way that extends their scalar counterparts. Although these names have been used other times in the existing literature (see, i.e., Sawaragi **et al**. (1985) and references therein), the approach presented here is quite different than the developments previously published.

Unfortunately, in our proposal the weak efficiency is not powerful enough to get directly a dual vector problem definition, even under convexity or linear hypothesis. However, we will see that this can be accomplished if we additionally impose that the outcome set recedes in every direction of the nonpositive orthant R^k_-.

This article is organized as follows: Section 2 presents some theoretical prerequisites that will be used later, including two unknown theorems of the alternative for the weakly efficient case. In Section 3 we give a classification of multiple objective programs according to the set of weakly efficient solutions. In Section 4 we introduce the notions of weakly efficient bounds and suprema of a MOP and we obtain some properties and results. The paper concludes with some final remarks.

2. THEORETICAL BACKGROUND

In this section we provide the reader with some previous results that will assist our developments. Particularly, the theorem of the alternative stated in Corollary 2.4 emerges as a highly useful tool for theoretical developments concerning the MOLP, as will be shown in section 4.

First, we start with a definition.

Definition 2.1 (Rockafellar (1970), p. 61) *Let* $X \subseteq R^n$ *be a non-empty set and* $d \in R^n - \{0\}$*. We shall say that* X *recedes in the direction of* d *if, and only if,* $\forall x \in X$*,* $\forall \beta \in R_+$*, we have* $x + \beta d \in X$*.*

Directions in which X recedes are referred to as directions of recession of X.

Let $\lambda \in R^k$. We define the program P_λ as:

$$\max\left\{\lambda^t z(x) / x \in X\right\} \tag{3}$$

In the sequel, S_{P_λ} and D_λ will denote, respectively, the set of all optimal solutions of problem P_λ and the dual problem of (3).

It is well known that, in the linear case, the following result holds:

Theorem 2.2 (Steuer (1986), Theorem 9.25) $\bar{x} \in WE^P$ *if, and only if,* $\exists \lambda \in R_+^k - \{0\}$ *such that* $\bar{x} \in S_{P_\lambda}$.

Let us consider an arbitrary $\alpha \in R^k$. Without loss of generality, we can assume $C \neq 0$. To the author's knowledge, the following theorems of the alternative appear in the literature for the first time.

Theorem 2.3 *The system*

$$Ax = b,\ x \geqq 0,\ Cx > \alpha \tag{4}$$

is feasible if, and only if, the system

$$u^t A - \lambda^t C \geqq 0^t,\ \lambda^t \alpha - \delta = u^t b,\ (\lambda^t,\ \ \delta) \geq 0 \tag{5}$$

is infeasible.

Proof. Clearly, system (4) being feasible is equivalent to system $-Ax + by = 0$, $y > 0$, $x \geqq 0$, $Cx - \alpha y > 0$, being feasible. Since this system can be rewritten as:

$$(-A,\ \ b)\begin{pmatrix} x \\ y \end{pmatrix} = 0,\ \begin{pmatrix} C & -\alpha \\ 0^t & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} > 0,\ (I,\ \ 0)\begin{pmatrix} x \\ y \end{pmatrix} \geqq 0, \tag{6}$$

applying Motzkin's theorem of the alternative (Mangasarian (1969), p. 28), system (6) is feasible if, and only if, system $(\lambda^t,\ \ \delta)\begin{pmatrix} C & -\alpha \\ 0^t & 1 \end{pmatrix} + s^t(I,\ \ 0) + u^t(-A,\ \ b) = 0$, $s \geqq 0$, $(\lambda^t,\ \ \delta) \geq 0$, is infeasible, which is equivalent to system (5) having no solution.

■

The above result can be improved under the assumption that $X = \left\{x \in R_+^n / Ax = b\right\}$ is not empty.

Corollary 2.4 *If* $X \neq \emptyset$ *then system (4) is infeasible if, and only if, the system*

$$u^t A - \lambda^t C \geqq 0^t,\ \lambda^t \alpha \geq u^t b,\ \lambda \geq 0 \tag{7}$$

is feasible.

Proof. By Theorem 2.3, system (4) is infeasible if, and only if, $u^t A - \lambda^t C \geqq 0^t$, $\lambda^t \alpha - \delta = u^t b$, $(\lambda^t,\ \ \delta) \geq 0$, is feasible. Now, suppose by contradiction that $\lambda = 0$, $\delta > 0$. Thus $u^t A \geqq 0^t$ and $u^t b < 0$. On the other hand, taking an arbitrary $x \in X$ we obtain $u^t Ax = u^t b \geq 0$, which is a contradiction. Therefore, system (7) is feasible.

■

3. A CLASSIFICATION OF MULTIPLE OBJECTIVE PROGRAMS

In this section we shall focus on establishing a classification, in a weak sense, of multiple-objective problems which is a generalization of that existing for scalar objective programs. The classification presented here is alike to other developed in Isermann (1976) for the linear efficient case.

Let P be a MOP given by (1). We will say that:

Definition 3.1 P *is infeasible if, and only if,* $X = \emptyset$.

Definition 3.2 P *is weakly bounded if, and only if,* $WE^P \neq \emptyset$. *Otherwise* P *is said to be weakly unbounded.*

Note that P can be weakly bounded and WE^P unbounded.

The following is a mutually exclusive and exhaustive classification of multiple objective programs.

Proposition 3.3 *Only one of the following assertions holds: (i)* P *is infeasible, (ii)* P *is weakly bounded, (iii)* P *is weakly unbounded.*

It can be shown (by means of Theorem 2.2) that, in the linear case, if X is bounded then P is weakly bounded. Of course, the converse implication is not necessarily satisfied, as illustrated in the following example.

Example 3.4 *Let* P *be the MOLP* $\max\{(-x_1, x_2) / x_2 \leq 1, x \in R^2_+\}$. *It is clear that* X *is unbounded (see Figure 1) and* $WE^P = \{(0, x_2) \in R^2_+ / x_2 \leq 1\} \cup \{(x_1, 1) \in R^2_+\}$.

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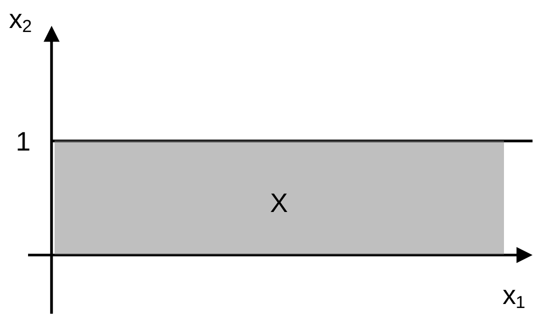

Figura 1.

4. THE CONCEPTS OF WEAKLY EFFICIENT BOUNDS AND SUPREMA

In this section we introduce the notions of weakly efficient bounds and suprema of a MOP, as generalizations of their scalar counterparts. This allows us to develop a theory that shows a connection, under certain conditions, between the weakly efficient suprema set of a CMP and its weakly nondominated solutions. In particular, a dual formulation, in a weakly efficient sense, for the MOLP is obtained.

Let P be an arbitrary MOP. The next concept plays a key role.

Definition 4.1 $\alpha \in R^k$ *is said to be a weakly efficient bound for* P *if, and only if, there exists no* $x \in X$ *such that* $z(x) > \alpha$.

Let WB(P) denote the set of all weakly efficient bounds for problem P.

Example 4.2 *Let us assume that* $z(X) = \{z \in R^2_+ / z_1 \leq 1, z_2 \leq 1\}$ *(see Figure 2 for a graphical representation). Then, it can be shown that* $WB(P) = \{\alpha \in R^2 / \alpha_1 \geq 1\} \cup \{\alpha \in R^2 / \alpha_2 \geq 1\}$.

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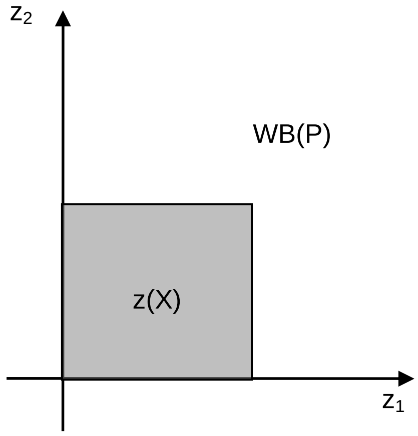

Figure 2.

Example 4.2 shows us that, in general, WB(P) is not convex (even whether z(X) is).

An interesting property of WB(P) is the following:

Proposition 4.3 WB(P) *is closed.*

Proof. Suppose that $\alpha^1, \alpha^2, \ldots$ is a sequence of elements of WB(P) that converges to some α^*. We need to show that $\alpha^* \in$ WB(P). Suppose on the contrary that $\alpha^* \notin$ WB(P). Then $\exists\, \bar{x} \in X$ such that $\bar{z} = z(\bar{x}) > \alpha^*$. Since $\lim_{k\to\infty} \alpha^k = \alpha^*$ we have $\exists\, k_0 \in N$, such that $\forall\, k \geq k_0$, $\alpha^k < \bar{z}$. Therefore, $\forall\, k \geq k_0$, $\alpha^k \notin WB(P)$, a contradiction. So, α^* belongs to WB(P).

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The following result states that every weakly nondominated point of z(X) is also a weakly efficient bound for P.

Proposition 4.4 $z(WE^P) \subseteq WB(P)$.

Proof. Let $\bar{z} \in z(WE^P)$. Then $\exists\, \bar{x} \in WE^P$ such that $z(\bar{x}) = \bar{z}$. Since by Definition 1.2 $\nexists\, x \in X$ verifying $z(x) > \bar{z}$ we conclude $\bar{z} \in WB(P)$.

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Of course, from the above proposition we get as direct consequences:

Corollary 4.5 *If* P *is weakly bounded then* $WB(P) \neq \varnothing$.

Corollary 4.6 $z(WE^P) = WB(P) \cap z(X)$.

Furthermore, it is clear:

Proposition 4.7 $\forall\, \alpha \in z(WE^P) + R^k_{--}$ *then* $\alpha \notin WB(P)$.

It is important to remark that, in the general case, P can be weakly unbounded and, however, $WB(P) \neq \varnothing$. For instance, let us consider the following example.

Example 4.8 *Let* $P \equiv \max\left\{(x_1, x_2) / x_2 + \frac{1}{2^{x_1}} \leq 0\right\}$ *(see Figure 3 for the graphical representation of its feasible region). Then,* $WB(P) = \{\alpha \in R^2 / \alpha_2 \geq 0\}$ and $WE^P = \varnothing$.

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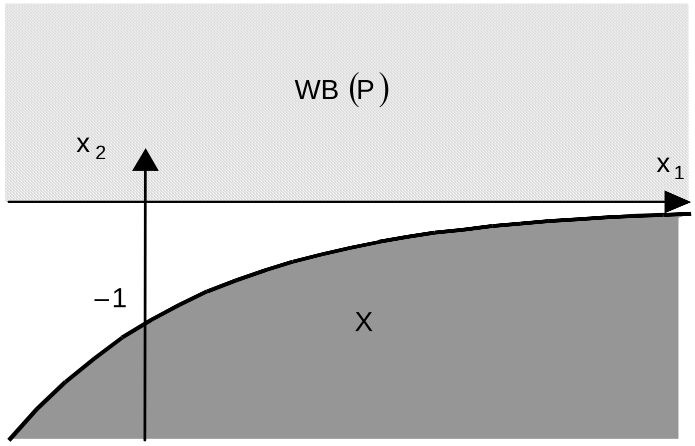

Figure 3.

The following theorem states an interesting property.

Theorem 4.9 *If* $X \neq \emptyset$ *and* P *is a MOLP as that given in (2), with* $WB(P) \neq \emptyset$, *then* P *is weakly bounded.*

Proof. If $WB(P) \neq \emptyset$ then $\exists\ \alpha \in R^k$ such that the system $Ax = b$, $x \geqq 0$, $Cx > \alpha$, is infeasible. Since $X \neq \emptyset$, by Corollary 2.4, it is equivalent to system $u^tA - \lambda^tC \geqq 0^t$, $\lambda^t\alpha \geq u^tb$, $\lambda \geq 0$, is feasible. Therefore, $\exists\, \lambda \in R^k_+ - \{0\}$ such that P_λ and D_λ are both feasible (remember that D_λ is the dual of P_λ). Thus $\exists\, \lambda \in R^k_+ - \{0\}$ such that P_λ is bounded. Now by Theorem 2.2 we conclude that $WE^P \neq \emptyset$.

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It seems clear that, the tighter the weakly efficient bounds the more useful they are. For this reason, we are interested in those weakly efficient bounds that are minimal in a weakly efficient vector sense.

Definition 4.10 *The elements of* $WB(P)$ *that are weakly nondominated by the minimum criterion, that is, the weakly efficient solutions of the following problem*

$$\min\{\alpha / \alpha \in WB(P)\}, \qquad (8)$$

are called weakly efficient suprema of P.

We will denote by WS(P) the set of all weakly efficient suprema of P.

Example 4.11 *Considering Example 4.8 again, we have* $WS(P) = \{x \in R^2 / x_2 = 0\}$.

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Example 4.12 *In the trivial case with* $X = \emptyset$ *then* $WB(P) = R^k$ *and thus* WS(P) = $\emptyset$.

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In the more general case, computing all the elements of WS(P) can be a very hard task because WS(P) is usually a nonconvex set.

The following result states that any weakly nondominated solution of P is always a weakly efficient supremum.

Theorem 4.13 $z(WE^P) \subseteq WS(P)$.

Proof. Let $\bar{z} \in z(WE^P)$. We know by Proposition 4.4 that $\bar{z} \in WB(P)$. Suppose on the contrary that $\bar{z} \notin WS(P)$. Then, by definition, $\exists\, \hat{z} \in WB(P)$ such that $\hat{z} < \bar{z}$. Since $\exists\, \bar{x} \in X$ verifying $\bar{z} = z(\bar{x}) > \hat{z}$ we have $\hat{z} \notin WB(P)$, a contradiction. Thus, it follows that $\bar{z} \in WS(P)$.

■

The next result follows immediately from Theorem 4.13:

Corollary 4.14 *If P is weakly bounded then* $WS(P) \neq \emptyset$.

The converse of this statement, however, does not hold. Indeed, it is possible that $WS(P) \neq \emptyset$ although P is weakly unbounded.

Example 4.15 *Let us consider* $z(X) = \{(z_1, z_2) \in R^2 / (z_1 z_2 \geq -1, z_1 < 0)\} \bigcup \{(z_1, z_2) \in R^2 / (z_1 z_2 \leq -1, z_1 > 0)\}$ *(see Figure 4 for a graphical representation). Then, it is clear that* $WE^P = \emptyset$, $WB(P) = R^2_+$ and $WS(P) = \{(0, \alpha) / \alpha \in R_+\} \bigcup \{(\alpha, 0) / \alpha \in R_+\}$.

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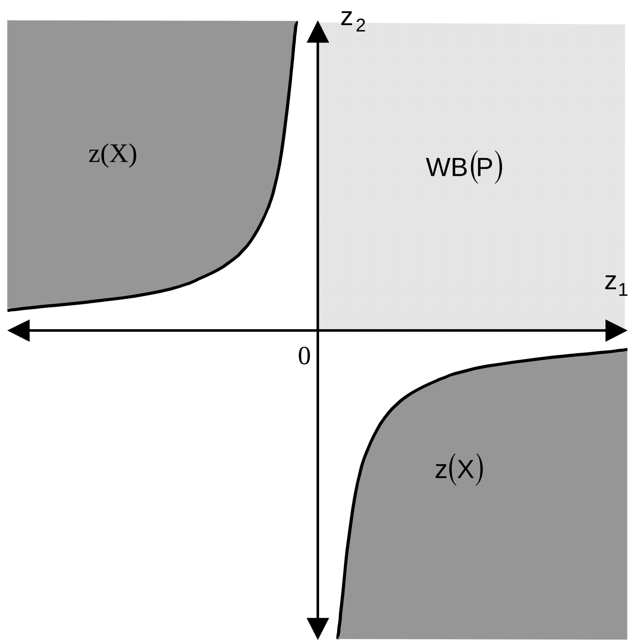

Figure 4.

We are going to find what conditions guarantee $WS(P) \subseteq z(X)$. First, it is clear that although $z(X)$ were closed and convex or even a polyhedral set, the above inclusion could be wrong. The following example proofs this:

Example 4.16 *Let* P *be the MOLP* $\max\{(x_1, x_2) / x_2 \leq 1, x \in R^2_+\}$. *It is clear that* $z(X) = X$ *is an unbounded polyhedron. Furthermore,* $WB(P) = \{\alpha \in R^2 / \alpha_2 \geq 1\}$, $WS(P) = \{\alpha \in R^2 / \alpha_2 = 1\}$ *and* $WS(P) \not\subseteq z(X)$.

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Now we will be concerned with establishing some conditions that guarantee the factibility of the weakly efficient suprema set. We need first a preliminary result:

Proposition 4.17 *Let* Y *be a nonempty subset of* R^k *and* $H \subseteq R^k$ *a hyperplane given by* $\{z \in R^k / \alpha^t z = \beta\}$. *If* Y *recedes in every direction of* R^k_- *and* $H \bigcap Y = \emptyset$ *then* α *has all its components of the same sign.*

Proof. Contrary to the conclusion, suppose that we can partitionate $\{1, ..., k\}$ in two nonempty sets, I and J such that $\alpha_I \geq 0$ and $\alpha_J \leq 0$. It is clear that we can get a vector $d \in R^k_{--}$ with the property $d^t \alpha = 0$. Therefore, d is a direction of recession of H. Thus, given $\bar{y} \in Y$ and $\hat{y} \in H$ we have that $\exists \lambda \in R_+$ verifying $\tilde{y} = \hat{y} + \lambda d \in H$ and $\tilde{y} < \bar{y}$. Since $Y = Y + R^k_-$ we obtain $\tilde{y} \in Y$. So, $H \bigcap Y \neq \emptyset$, a contradiction.

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Theorem 4.18 *Let* $z(X)$ *be nonempty, closed and convex. If* $z(X)$ *recedes in every direction of* R^k_- *then* $WS(P) \subseteq z(X)$.

Proof. Let $\bar{z} \in WS(P)$. Contrary to the conclusion, assume that $\bar{z} \notin z(X)$. Then, there exists a hyperplane $H = \{z \in R^k / \alpha^t z = \beta\}$ such that $\alpha^t z < \beta$, $\forall z \in z(X)$ and $\alpha^t \bar{z} > \beta$. Since, $H \cap z(X) = \varnothing$ and $z(X) = z(X) + R^k_-$, by Proposition 4.17 we know that α has all its components of the same sign. Now, having in mind that $\alpha^t z < \beta$, $\forall z \in z(X)$ we have $\alpha \in R^k_+$. Let $\gamma \in R_{++}$ and $\tilde{z} = \bar{z} - \gamma e$ such that $\alpha^t \tilde{z} > \beta$. It is clear that $\tilde{z} < \bar{z}$. Suppose on the contrary that $\exists\, x \in X$, $z(x) > \tilde{z}$. This implies that $\alpha^t z(x) > \alpha^t \tilde{z} > \beta$, a contradiction, so $\tilde{z} \in WB(P)$. Since $\tilde{z} < \bar{z} \Rightarrow \bar{z} \notin WS(P)$. This contradicts the initial hypothesis. Therefore, $\bar{z} \in z(X)$.

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Corollary 4.19 *Let* $z(X)$ *be nonempty, closed, convex and such that it recedes in every direction of* R^k_-*. Then* $WS(P) \subseteq z(WE^P)$.

Proof. *Let* $\bar{z} \in WS(P)$. By Theorem 4.18 we have that $\bar{z} \in z(X)$. Since, $\bar{z} \in WB(P)$, applying Corollary 4.6, $\bar{z} \in z(WE^P)$.

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Corollary 4.20 *Let* $z(X)$ *be nonempty, closed, convex and such that it recedes in every direction of* R^k_-*. Then* $WS(P) = z(WE^P)$.

Proof. Follows immediately from Theorem 4.13 and Corollary 4.19.

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The next results are direct consequences of Corollary 4.20.

Corollary 4.21 *If* P *is a CMP such that* $z(X)$ *recedes in every direction of* R^k_- *then* $WS(P) = z(WE^P)$.

Corollary 4.22 *If* P *is a MOLP such that* $z(X)$ *recedes in every direction of* R^k_- *then* $WS(P) = z(WE^P)$.

The above results are very important in order to develop a duality theory for the convex and linear cases under the assumption that $z(X) \neq \varnothing$ recedes in every direction of R^k_-. For these types of problems, to compute their weakly nondominated solutions is equivalent to solve D $\equiv$ $\min\{\alpha / \alpha \in WB(P)\}$, in a weakly efficient vector sense, due to the fact that $z(WE^P) = WE^D$.

Moreover, for a linear problem P, $WB(P) = \{\alpha \in R^k / Ax = b, x \geqq 0, Cx > \alpha, \text{has no solution}\}$. Applying Corollary 2.4, we conclude that D $\equiv$ $\min\{\alpha \in R^k / u^t A \geqq \lambda^t C, u^t b \leq \lambda^t \alpha, \lambda \geq 0, u \in R^m\}$. Thus, we have a dual formulation, in a weakly efficient sense, for the MOLP problem P.

5. CONCLUSIONS

We have derived some new theorems of the alternative which enable us with powerful tools for theoretical developments in vector optimization involving weakly efficient solutions. Actually, such theorems seem to have applications in a wide variety of contexts (see, e.g., Jorge (2002a)).

Also, the notions of weakly efficient bounds and suprema for a MOP problem were introduced. Such concepts have an obvious geometric meaning that allows an easy interpretation in the criteria space, R^k.

The developed theory establishes a sufficient condition under which it is possible to define a dual vector problem for a CMP in a weakly efficient sense. With these hypotheses, a restricted dual vector problem definition for the MOLP, similar to that given in Gale et al. (1951), is achieved.

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