STEADY STATE ANALYSIS OF A SINGLE SERVER BULK QUEUE WITH GENERAL VACATION TIMES AND RESTRICTED ADMISSIBILITY OF ARRIVING BATCHES

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ABSTRACT
We analyze the steady state behavior of a single server vacation queue with variable batch size arrivals in a compound Poisson Process, exponential service in batch of fixed size b following a min(b,n) rule and general server vacations. However, an arriving batch may or may not be allowed to join the system at all the times. We obtain explicit steady state results for the probability generating functions for the number of customers in the system, the average number of customers and the average waiting time in the queue and the system. Some special cases of interest have been derived and finally a numerical example is provided.

Key words: Compond Poisson process, restricted admissibility, bulk queues, Bernoulli schedule.

MSC: 60K25

RESUMEN
Analizamos el comportamiento del estado de la cola de la vacación de un servidor sencillo con tamaño del lote variable en un proceso de Poisson compuesto, el servicio exponencial en lotes de tamaño fijo b siguiendo una regla min(b,n) con servidor general de vacaciones. Sin embargo, puede permitirse o no incluir en el sistema un lote que arribe en todos los momentos. Obtenemos expresiones explícitas de la estabilidad del estado para las funciones generatrices de probabilidad para el número de clientes en el sistema, el número promedio de clientes y el promedio del tiempo de espera en la cola y en el sistema. Algunos casos de interés especiales han sido derivados y finalmente un ejemplo numérico es desarrollado.

1. INTRODUCTION

There is extensive literature on bulk queues, for example, see Jaiswal [1960], Bhat [1964], Medhi [1975], Chaudhry and Templeton [1983] and Neuts [1987], to mention a few. In this paper, we study the steady state behavior of a bulk queue with Bernoulli schedule server vacations. Vacation queues with Bernoulli schedule and many other vacation policies have been widely studied by numerous authors including Scholl and Kleinrock [1983], Keilson and Servi [1986], Shanthikumar [1988], Cramer [1989], Madan [1992, 1999, 2001] and Madan and Saleh [2001]. For a complete overview of queues with vacations, see Doshi [1986]. In the present paper we use supplementary variable technique to study a single server bulk queue with Bernoulli schedule server vacations from a different standpoint. Our additional key assumption is that not all-arriving batches are allowed into the system at all times. We further assume different policies regarding admissibility of batches for the period when the server is present in the system and for the period of server vacations. In the case of a mechanical server, the breakdown periods correspond to the vacation periods of the human server. One may encounter such queueing situations on some traffic highways, supermarkets, airports and some communication and computer systems where the management may have to decide to adopt a policy of restricting the input into the system from time to time. Having thus turned away, an arriving batch immediately leaves the system and is lost to the system. Our mathematical model is briefly described by the following assumptions:

2. ASSUMPTIONS UNDERLYING THE MATHEMATICAL MODEL

1. Customers arrive at the system in batches of variable size according to a compound Poisson process with arrival rate $\lambda\,(>0)$. Let π_i be the probability that a batch of size i arrives at the system where $\sum_{i=1}^{\infty}\pi_i = 1$.

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2. Not all-arriving batches are allowed to join the system at all the times. Let r_1 $(0 < r_1 \le 1)$ be the probability that an arriving batch will be allowed to join the system while server is working and let r_2 $(0 \le r_2 \le 1)$ be the probability that an arriving batch will be allowed to join the system during servers vacation period.

3. Service to customers is provided in batches of fixed size b following a min (b, n) rule which means that a fixed number b of customers or the entire queue length, whichever is less, is taken up for service. The service time of a batch of costomers is assumed to be negative exponential with mean $1/\mu$. We further assume, without loss of generality, that the customers in a batch are pre-ordered for service.

4. We assume Bernoulli schedule server vacations, which means that the server can take a vacation only at the time marks of completion of a service. At such an instant, the server may take a vacation with probability p and may not take a vacation with probability 1-p.

5. The vacation times follow a general distribution with probability density function b(v) and the distribution function B(v) where v is the vacation time of the server. Let $\xi(x)\,dx$ be the first order probability that the vacation of the server will complete during the interval (x, x+dx) given that the same was not complete till time x. Therefore,

$$\xi(x) = \frac{b(x)}{1 - B(x)} \tag{1}$$

So that

$$b(v) = \xi(v) \exp\left[- \int_0^v \xi(x) dx \right] \tag{2}$$

6. Various stochastic processes involved in the system are independent of each other.

3. DEFINITIONS AND EQUATIONS

We define:

$P_n(t)$: probability that at time t the server is providing service and there are $n\,(\ge 0)$ customers in the system including a batch in service, if any. However, if n = 0 at time t, this means that the server is present in the system but he is idle at such instants.

$V_n(x,t)$: probability that at time t there are $n\,(\ge 0)$ customers in the system and the server is on vacation with elapsed vacation time lying between x and x + dx.

Correspondingly $V_n(t) = \int_0^\infty V_n(x,t)\,dx$ is the probability that at time t there are $n\,(\ge 0)$ customers in the system and the server is on vacation irrespective of the elapsed vacation time x.

Then, assuming that the steady state exists, we let $\lim_{t\to\infty} P_n(t) = P_n$, $\lim_{t\to\infty} V_n(x,t) = V_n(x)$, $\lim_{t\to\infty} V_n(t) = V_n$ so that P_n, $V_n(x)$ and V_n denote the respective steady state probabilities.

Further we define the following probability generating functions:

$$P(z) = \sum_{n=0}^{\infty} P_n z^n \quad , \tag{3a}$$

$$V(z) = \sum_{n=0}^{\infty} V_n z^n \, , \tag{3b}$$

$$\pi(z) = \sum_{i=1}^{\infty} \pi_i z^i , \qquad |z| \le 1 . \tag{3c}$$

4. STEADY STATE EQUATIONS

Connecting states of the system at time t + dt with those at time t, using usual probability reasoning and then taking limit as $t \to \infty$, etc, we have the following set of differential difference equations:

$$(\lambda + \mu)P_n = \sum_{i=1}^{n} \lambda\pi_i r_1 P_{n-i} + \sum_{i=1}^{\infty} \lambda\pi_i (1-r_1)P_n + \mu(1-p)\sum_{j=1}^{b} P_{n+j} + \int_0^{\infty} V_n(x)\xi(x)dx , n \ge 1 \tag{4}$$

$$\lambda P_0 = \sum_{i=1}^{\infty} \lambda\pi_i (1-r_1)P_0 + \mu(1-p)\sum_{j=1}^{b} P_j + \int_0^{\infty} V_0(x)\xi(x)dx , \quad n = 0 \tag{5}$$

$$\frac{\partial}{\partial x}V_n(x) + \left(\lambda + \xi(x)\right)V_n(x) = \sum_{i=1}^{n} \lambda\pi_i r_2 V_{n-i}(x) + \sum_{i=1}^{\infty} \lambda\pi_i (1-r_2)V_n(x) , \qquad n \ge 1 \tag{6}$$

$$\frac{\partial}{\partial x}V_0(x) + \left(\lambda + \xi(x)\right)V_0(x) = \sum_{i=1}^{\infty} \lambda\pi_i (1-r_2)V_0(x) , \qquad n = 0 . \tag{7}$$

The above equations (4) to (7) are to be solved subject to the boundary conditions

$$V_n(0)+ = \mu\, p\sum_{j=1}^{b} P_{n+j} , \; n \ge 0 . \tag{8}$$

5. STEADY STATE PROBABILITY GENERATING FUNCTIONS

We multiply both sides of equation (4) by z^{n+b} and sum over n from 1 to ∞. Then we multiply both sides of (5) by z^b and add the two results, simplify and use equations (3a), (3b) and (3c). We thus have

$$(\lambda + \mu)z^b P(z) - \mu z^b P_0 = \lambda r_1 \pi(z)P(z)z^b + \mu(1-p)\sum_{j=1}^{b-1} z^j P(z) - \mu(1-p)\sum_{k=0}^{b-1} \sum_{j=k}^{b-1} z^j P_k$$
$$+ \lambda(1-r_1)z^b P(z) + z^b \int_0^{\infty} V(x,z)\xi(x)dx , \tag{9}$$

which simplifies to

$$\left[\left(\lambda r_1 - \lambda r_1 \pi(z) + \mu\right)z^b - \mu(1-p)\sum_{j=1}^{b-1} z^j\right]P(z) = z^b \int_0^{\infty} V(x,z)\xi(x)dx + \mu z^b P_0 - \mu(1-p)\sum_{k=0}^{b-1} \sum_{j=k}^{b-1} z^j P_k . \tag{9a}$$

Again, we multiply both sides of equation (6) by z^n sum over all n from 1 to ∞ and add the result to equation (7). Then on simplifying and using (3b), (3c) we have

$$\frac{\partial}{\partial x} V(x, z) + \left[\lambda r_2 - \lambda r_2 \pi(z) + \xi(x)\right]V(x, z) = 0 . \tag{10}$$

Next, we multiply both sides of equation (8) by z^{n+b}, sum over n from 1 to ∞, use (3a), (3b) and simplify. We thus have

$$z^b V(0,z) = \mu p\sum_{j=0}^{b-1} z^j P(z) - \mu p\sum_{k=0}^{b-1} \sum_{j=k}^{b-1} z^j P_k . \tag{11}$$

Now, we Integrate (10) between the limits 0 and x, we have

$$V(x,z) = V(0,z)\exp\left[-\left[\lambda r_2\left(1-\pi(z)\right)\right]x - \int_0^x \xi(t)dt\right]. \tag{12}$$

Next, we integrate (12) by parts w. r. t . x and use (2). We then have

$$V(z) = \frac{\mu p}{z^b}\left[\sum_{j=0}^{b-1} z^j P(z) - \sum_{k=0}^{b-1}\sum_{j=k}^{b-1} z^j P_k\right]\left[\frac{1-\bar{b}\left(\lambda r_2 - \lambda r_2\pi(z)\right)}{\lambda r_2 - \lambda r_2 \pi(z)}\right], \tag{13}$$

where $\bar{b}\left[\lambda r_2 - \lambda r_2\pi(z)\right] = \int_0^\infty \exp\left[-\left(\lambda r_2 - \lambda r_2\pi(z)\right)x\right]b(x)dx$ is the laplace transform of b(x).

Now, we consider the integral $\int_0^\infty V(x,z)\xi(x)dx$ which appears in equation (9a). Substituting for V(x,z) from equation (12) this integral becomes

$$\int_0^\infty V(x,z)\xi(x)dx = V(0,z)\int_0^\infty \exp\left[-\left(\lambda r_2 - \lambda r_2\pi(z)\right)x - \int_0^x \xi(t)dt\right]\xi(x)dx, \tag{14}$$

which on using (2) yields

$$\int_0^\infty V(x,z)\xi(x)dx = V(0,z)\,\bar{b}\left(\lambda r_2 - \lambda r_2\pi(z)\right). \tag{15}$$

Using (15) into (9), we obtain

$$\left[\left(\lambda r_1 - \lambda r_1\pi(z) + \mu\right)z^b - \mu(1-p)\sum_{j=0}^{b-1} z^j\right]P(z) = z^b V(0,z)\bar{b}\left(\lambda r_2 - \lambda r_2\pi(z)\right) + \mu z^b P_0 - \mu(1-p)\sum_{k=0}^{b-1}\sum_{j=k}^{b-1} z^j P_k \tag{16}$$

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Substituting for $z^b V(0,z)$ from equation (11) into equation (16), we obtain

$$\left[\left(\lambda r_1 - \lambda r_1\pi(z) + \mu\right)z^b - \mu(1-p)\sum_{j=0}^{b-1} z^j\right]P(z) = \mu p\left[\bar{b}\left(\lambda r_2 - \lambda r_2\pi(z)\right)\right]\sum_{j=0}^{b-1} z^j P(z)$$

$$-\mu p\left[\bar{b}\left(\lambda r_2 - \lambda r_2\pi(z)\right)\right]\sum_{k=0}^{b-1}\sum_{j=k}^{b-1} z^j P_k + \mu z^b P_0 - \mu(1-p)\sum_{k=0}^{b-1}\sum_{j=k}^{b-1} z^j P_k \tag{17}$$

which simplifies to

$$\left[\left(\lambda r_1 - \lambda r_1\pi(z) + \mu\right)z^b - \mu(1-p)\sum_{j=0}^{b-1} z^j - \mu p\left[\bar{b}\left(\lambda r_2 - \lambda r_2\pi(z)\right)\right]\sum_{j=0}^{b-1} z^j\right]P(z)$$

$$= \mu z^b P_0 - \mu\sum_{k=0}^{b-1}\sum_{j=k}^{b-1} z^j P_k\left[(1-p) + p\bar{b}\left(\lambda r_2 - \lambda r_2\pi(z)\right)\right], \tag{17a}$$

where

$$P(z) = \frac{\mu z^b P_0 - \mu\sum_{k=0}^{b-1}\sum_{j=k}^{b-1} z^j P_k\left[(1-p) + p\bar{b}\left(\lambda r_2 - \lambda r_2\pi(z)\right)\right]}{\left[\left(\lambda r_1 - \lambda r_1\pi(z) + \mu\right)z^b - \mu(1-p)\sum_{j=0}^{b-1} z^j - \mu p\left[\bar{b}\left(\lambda r_2 - \lambda r_2\pi(z)\right)\right]\sum_{j=0}^{b-1} z^j\right]}. \tag{18}$$

Using (18) into (13), V (z) can also be obtained.

It can be shown by Roche's theorem that the denommeter of the RHS of (18) has b zeroes inside the unit circle $|z| = 1$ which are sufficient to determine all the b unknowns $P_k, k = 0,1,2,3,\ldots,(b-1)$ which appear in the numerator of P (z) in (18).

6. SPECIAL CASES

Case 1: All ARRIVING BATCHES ARE ALLOWED INTO THE SYSTEM AT ALL TIMES

In this case, we let $r_1 = r_2 = 1$ into the main results found in equations (18) and (13). We then obtain

$$P(z) = \frac{\mu z^b P_0 - \mu \sum_{k=0}^{b-1} \sum_{j=k}^{b-1} z^j P_k \left[(1-p) + p\bar{b}(\lambda - \lambda\pi(z))\right]}{\left[(\lambda - \lambda\pi(z) + \mu) z^b - \mu(1-p)\sum_{j=0}^{b-1} z^j - \mu p\left[\bar{b}(\lambda - \lambda\pi(z))\right]\sum_{j=0}^{b-1} z^j\right]}, \tag{19}$$

$$V(z) = \frac{\mu p}{z^b}\left[\sum_{j=0}^{b-1} z^j P(z) - \sum_{k=0}^{b-1} \sum_{j=k}^{b-1} z^j P_k\right]\left[\frac{1-\bar{b}(\lambda - \lambda\pi(z))}{\lambda - \lambda\pi(z)}\right]. \tag{20}$$

Case 2: NO ARRIVALS DURING VACATIONS

In this case we let $r_1 = 1$ and $r_2 = 0$ in (18) and (13) and have

$$P(z) = \frac{\mu z^b P_0 - \mu \sum_{k=0}^{b-1} \sum_{j=k}^{b-1} z^j P_k \left[(1-p) + p\bar{b}(\lambda r_2 - \lambda r_2\pi(z))\right]}{\left[(\lambda - \lambda\pi(z) + \mu) z^b - \mu(1-p)\sum_{j=0}^{b-1} z^j - \mu p\bar{b}(\lambda r_2 - \lambda r_2\pi(z))\sum_{j=0}^{b-1} z^j\right]}, \tag{21}$$

$$V(z) = \frac{\mu p}{z^b}\left[\sum_{j=0}^{b-1} z^j P(z) - \sum_{k=0}^{b-1} \sum_{j=k}^{b-1} z^j P_k\right]\lim_{r_2\to 0}\left[\frac{1-\bar{b}(\lambda r_2 - \lambda r_2\pi(z))}{\lambda r_2 - \lambda r_2\pi(z)}\right]$$

$$= \frac{\mu p}{z^b}\left[\sum_{j=0}^{b-1} z^j P(z) - \sum_{k=0}^{b-1} \sum_{j=k}^{b-1} z^j P_k\right]\left(\frac{1}{\xi}\right). \tag{22}$$

It may be noted that we employed L'Hôpital's rule to obtain (22), since the second factor is indeterminate of the zero/zero from at $r_2 = 0$. Or alternatively (22) can also be obtained on letting $r_2 = 0$ in equation (12).

Case 3: NO VACATIONS

In this case we let p = 0 in equation (13) and (18) and have V(z) = 0 and

$$P(z) = \frac{\mu z^b P_0 - \mu \sum_{k=0}^{b-1} \sum_{j=k}^{b-1} z^j P_k}{(\lambda r_1 - \lambda r_1\pi(z) + \mu) z^b - \mu \sum_{j=1}^{b-1} z^j}. \tag{23}$$

Case 4: ONE BY ONE ARRIVALS, ONE BY ONE EXPONENTIAL SERVICE AND EXPONENTIAL VACATIONS

In this case, we have $\pi_1 = 1, \pi_i = 0$ for $i \neq 1, b = 1,\ \pi(z) = z$ and

$$\bar{b}[\lambda r_2 - \lambda r_2 \pi(z)] = \bar{b}[\lambda r_2 - \lambda r_2 z] = \frac{\xi}{\lambda r_2 - \lambda r_2 z + \xi}.$$

With these substations, equations (13) and (18) yield

$$V(z) = \frac{\mu p}{z}[P(z) - P_0]\left[\frac{1 - \dfrac{\xi}{\lambda r_2 - \lambda r_2 z + \xi}}{\lambda r_2 - \lambda r_2 z}\right], \tag{24}$$

$$P(z) = \frac{\mu z P_0 - \mu P_0\left[(1-p) + p\left(\dfrac{\xi}{\lambda r_2 - \lambda r_2 z + \xi}\right)\right]}{\left[(\lambda r_1 - \lambda r_1 z + \mu)z - \mu(1-p) - \mu p\left(\dfrac{\xi}{\lambda r_2 - \lambda r_2 z + \xi}\right)\right]}. \tag{25}$$

We can further simplify equations (24) and (25) as

$$V(z) = \frac{\mu p}{z}[P(z) - P_0]\left[\frac{1}{\lambda r_2 - \lambda r_2 z + \xi}\right], \tag{24a}$$

$$P(z) = \frac{\mu P_0[z(\lambda r_2 - \lambda r_2 z + \xi) - (1-p)(\lambda r_2 - \lambda r_2 z + \xi) - p\xi]}{(\lambda r_2 - \lambda r_2 z + \xi)(\lambda r_1 - \lambda r_1 z + \mu)z - \mu(1-p)(\lambda r_2 - \lambda r_2 z + \xi) - \mu p \xi}. \tag{25a}$$

Now, we have to determine the only unknown P_0 which appears in the numerators of (24a) and (25a). Using L'H$\hat{o}$pital's rule (25a) yields

$$P(1) = \lim_{z\to 1} P(z) = \frac{\mu(\xi - \lambda r_2 p)P_0}{\mu\,\xi - \lambda\xi\, r_1 - \mu\lambda p r_2}. \tag{26}$$

Then, using (26), equation (24a) yields

$$V(1) = \lim_{z\to 1} V(z) = \lim_{z\to 1}\frac{\mu p}{z}[P(z) - P_0]\left[\frac{1}{\lambda r_2 - \lambda r_2 z + \xi}\right] = \frac{\mu p}{\xi}\left[\lim_{z\to 1} P(z) - P_0\right] = \frac{\mu p}{\xi}[P(1) - P_0] = \frac{\mu\lambda r_1 p P_0}{\mu\,\xi - \lambda\xi\, r_1 - \mu\lambda p r_2}. \tag{27}$$

Using (26) and (27) in the normalizing condition P(1) + V(1) = 1, we obtain

$$P_0 = \frac{\mu\,\xi - \lambda\xi r_1 - \mu\lambda p r_2}{\mu\,\xi + \mu\lambda p\, r_1 - \mu\lambda p r_2}, \tag{28}$$

provided

$$\lambda\xi r_1 + \mu\lambda p r_2 < \mu\,\xi. \tag{29}$$

In (29), we have the stability condition under which the steady state exists.

We note that when p = 0 and $r_1 = 1$ (29) reduces to $\frac{\lambda}{\mu} < 1$ which is the well known stability condition of the M/M/1 queueing system.

Further, substituting for P_0 from (28) into (26) and (27), we have

$$P = P(1) = \frac{\mu(\xi - \lambda r_2 p)}{\mu\,\xi + \mu\lambda p\, r_1 - \mu\lambda p r_2}\,, \tag{30}$$

$$V = V(1) = \frac{\mu\lambda r_1 p}{\mu\,\xi + \mu\lambda p\, r_1 - \mu\lambda p r_2}\,. \tag{31}$$

We note that equations (30) and (31) respectively give the steady state probabilities that the server is present in the system and he is on vacation.

We further note that P found in equation (30) is the proportion of time the server remains present in the system. Since this also includes the proportion of server's idle time, therefore, system's utilization factor ρ is given by

$$\rho = P - P_0 = \frac{\lambda\xi r_1}{\mu\xi + \mu\lambda p r_1 - \mu\lambda p r_2}\,. \tag{32}$$

We may further verify that when $r_1 = r_2 = 1$, equation (32) yields $\rho = \frac{\lambda}{\mu}$ which is the utilization factor for the M/M/1 queue.

Having found the unknown P_0 in (28), the probability generating functions obtained in (24a) and (25a) are now completely determined.

7. THE AVERAGE NUMBER AND THE AVERAGE WAITING TIME IN THE QUEUE AND THE SYSTEM

We shall find the average number and the average waiting time in the queue and the system only for the simplest particular case 4 where the arrivals are Poisson one by one, service is exponential one by one and also vacations are exponential. In this case, we define $T(z) = P(z) + V(z)$ to be the steady state probability generating function of the number in the system irrespective of whether the server is working or on vacation, where P(z) and V(z) are given in (25a) and (24a) respectively.

Substituting for P(z) from (25a) into (24a) we have on simplifying

$$V(z) = \frac{\mu\lambda p r_1 (z-1) P_0}{(\lambda r_2 - \lambda r_2 z + \xi)(\lambda r_1 - \lambda r_1 z + \mu) z - \mu(1-p)(\lambda r_2 - \lambda r_2 z + \xi) - p\mu\xi}\,, \tag{33}$$

where P_0 has already been found in (28). Then, we have on adding (25a) and (33),

$$T(z) = P(z) + V(z) = \frac{N(z)}{D(z)} \tag{34}$$

where

$$N(z) = \mu P_0\left[(\lambda r_2 - \lambda r_2 z + \xi) z - (1-p)(\lambda r_2 - \lambda r_2 z + \xi) - p\xi\right] + \mu p \lambda r_1 (z-1) P_0\,, \tag{35}$$

$$D(z) = (\lambda r_2 - \lambda r_2 z + \xi)(\lambda r_1 - \lambda r_1 z + \mu) z - \mu(1-p)(\lambda r_2 - \lambda r_2 z + \xi) - p\mu\xi\,. \tag{36}$$

However, since T(z) is indeterminate of the zero/zero form at z = 1, we employ L'Hôpital's rule twice and obtain

$$L = \lim_{z \to 1} \frac{N(z)}{D(z)} = \frac{D'(1)N''(1) - D''(1)N'(1)}{2\left[D'(1)\right]^2}\,, \tag{37}$$

where dashes denote derivatives w.r.t. at z = 1. After a lot of algebra and simplification, we have

$$N'(z) = \mu P_0\left[\xi + p\lambda r_1 - p\lambda r_2\right] \text{ and } N''(z) = -2\lambda\mu r_2 P_0\,, \tag{38}$$

$$D'(z) = \mu\xi - \lambda r_1\xi - p\lambda\mu r_2 \text{ and } D''(z) = 2\lambda\left[\lambda r_1 r_2 - r_1\xi - r_2\mu\right]. \tag{39}$$

Substituting (38) and (39) into (37), we have

$$L = \frac{\left[\mu\xi - \lambda r_1\xi - p\mu\lambda r_2\right]\left[-2\mu\lambda r_2 P_0\right] - \mu P_0\left[\xi + p\lambda r_1 - p\lambda r_2\right]\left[2\lambda\left(\lambda r_1 r_2 - r_1\xi - r_2\mu\right)\right]}{2\left[\mu\xi - \lambda r_1\xi - p\lambda\mu r_2\right]^2}. \tag{40}$$

Using Little's formulas, we further obtain L_q, the average number in the queue as

$$L_q = L - \rho, \tag{41}$$

where ρ has been found in equation (32).

Further, we shall find the average waiting time in the system, W and the average waiting time in the queue, W_q by again using Little's formulas as

$$W = \frac{L}{\lambda_a} \text{ and } W_q = \frac{L_q}{\lambda_a} \tag{42}$$

where λ_a is the actual arrival rate into the system. To find λ_a we note that we have obtained P, the proportion of times the server is present in the system and V, the proportion of times the server is on vacation in equations (30) and (31). Therefore, the actual arrival rate is given by

$$\lambda_a = \lambda r_1 P + \lambda r_2 V = \frac{\mu\lambda r_1\xi}{\mu\xi + \mu\lambda p r_1 - \mu\lambda p r_2}. \tag{43}$$

We note that when all arrivals are allowed to join the system at all times, then letting $r_1 = r_2 = 1$ in equation (43) yields actual arrival rate $\lambda_a = \lambda$ as it should be.

Finally substituting for L, L_q and λ_a from equations (40), (41) and (43), equation (42) explicitly yields W and W_q.

8. A NUMERICAL EXAMPLE

In order to see the effect of various parameters namely p, r_1 and r_2 on various characteristics such as server's idle time, server's vacation time, systems utilization factor, average number of customers and the average waiting time in the queue and in the system, we arbitrarily choose values of $\lambda = 5$, $\mu = 10$, $\xi = 15$ but vary the values of p, r_1 and r_2 from 0.0 to 1.0 such that the steady state condition found in equation (29) is always satisfied. Based on our results obtained in equations (30)-(32) and (40)-(43), the following tables give the computed values of the desired queue characteristics.

Table 1.
Computed values of various states of the system.

$\lambda = 5$, $\mu = 10$, $\zeta = 15$							
p	r_1	r_2	ρ	P_0	P	V	λ_a
0.0	0.4	0.0	0.200000	0.80000	1.00000	0.000000	2.00000
		0.2	0.200000	0.80000	1.00000	0.000000	2.00000
		0.4	0.200000	0.80000	1.00000	0.000000	2.00000
		0.6	0.200000	0.80000	1.00000	0.000000	2.00000
		0.8	0.200000	0.80000	1.00000	0.000000	2.00000
		1.0	0.200000	0.80000	1.00000	0.000000	2.00000
	1.0	0.0	0.500000	0.50000	1.00000	0.000000	5.00000
		0.2	0.500000	0.50000	1.00000	0.000000	5.00000
		0.4	0.500000	0.50000	1.00000	0.000000	5.00000
		0.6	0.500000	0.50000	1.00000	0.000000	5.00000
		0.8	0.500000	0.50000	1.00000	0.000000	5.00000
		1.0	0.500000	0.50000	1.00000	0.000000	5.00000
0.4	0.4	0.0	0.189873	0.75949	0.94937	0.075949	1.89873
		0.2	0.194805	0.75325	0.94805	0.077922	1.94805
		0.4	0.200000	0.74667	0.94667	0.080000	2.00000
		0.6	0.205479	0.73973	0.94521	0.082192	2.05479
		0.8	0.211268	0.73239	0.94366	0.084507	2.11268
		1.0	0.217391	0.72464	0.94203	0.086957	2.17391
	1.0	0.0	0.441176	0.44118	0.88235	0.176471	4.41176
		0.2	0.451807	0.42771	0.87952	0.180723	4.51807
		0.4	0.462963	0.41358	0.87654	0.185185	4.62963
		0.6	0.474684	0.39873	0.87342	0.189873	4.74684
		0.8	0.487013	0.38312	0.87013	0.194805	4.87013
		1.0	0.500000	0.36667	0.86667	0.200000	5.00000
1.0	0.4	0.0	0.176471	0.70588	0.88235	0.176471	1.76471
		0.2	0.187500	0.68750	0.87500	0.187500	1.87500
		0.4	0.200000	0.66667	0.86667	0.200000	2.00000
		0.6	0.214286	0.64286	0.85714	0.214286	2.14286
		0.8	0.230769	0.61538	0.84615	0.230769	2.30769
		1.0	0.250000	0.58333	0.83333	0.250000	2.50000
	1.0	0.0	0.375000	0.37500	0.75000	0.375000	3.75000
		0.2	0.394737	0.34211	0.73684	0.394737	3.94737
		0.4	0.416667	0.30556	0.72222	0.416667	4.16667
		0.6	0.441176	0.26471	0.70588	0.441176	4.41176
		0.8	0.468750	0.21875	0.68750	0.468750	4.68750
		1.0	0.500000	0.16667	0.66667	0.500000	5.00000

Table 2.
Computed values of various queue characteristics.

$\lambda = 5$, $\mu = 10$, $\zeta = 15$						
p	r_1	r_2	L	L_q	W	W_q
0.0	0.4	0.0	0.25000	0.05000	0.125000	0.025000
		0.2	0.25000	0.05000	0.125000	0.025000
		0.4	0.25000	0.05000	0.125000	0.025000
		0.6	0.25000	0.05000	0.125000	0.025000
		0.8	0.25000	0.05000	0.125000	0.025000
		1.0	0.25000	0.05000	0.125000	0.025000
	1.0	0.0	1.00000	0.50000	0.200000	0.100000
		0.2	1.00000	0.50000	0.200000	0.100000
		0.4	1.00000	0.50000	0.200000	0.100000
		0.6	1.00000	0.50000	0.200000	0.100000
		0.8	1.00000	0.50000	0.200000	0.100000
		1.0	1.00000	0.50000	0.200000	0.100000
0.4	0.4	0.0	0.25000	0.06013	0.131667	0.031667
		0.2	0.26265	0.06785	0.134828	0.034828
		0.4	0.27738	0.07738	0.138690	0.038690
		0.6	0.29452	0.08904	0.143333	0.043333
		0.8	0.31446	0.10320	0.148846	0.048846
		1.0	0.33768	0.12029	0.155333	0.055333
	1.0	0.0	1.00000	0.60526	0.253333	0.153333
		0.2	1.13908	0.72700	0.276418	0.176418
		0.4	1.32574	0.89470	0.307571	0.207571
		0.6	1.58398	1.13218	0.350588	0.250588
		0.8	1.95614	1.48145	0.412093	0.312093
		1.0	2.52381	2.02381	0.504762	0.404762
1.0	0.4	0.0	0.25000	0.07353	0.141667	0.041667
		0.2	0.28295	0.09545	0.150909	0.050909
		0.4	0.32667	0.12667	0.163333	0.063333
		0.6	0.38571	0.17143	0.180000	0.080000
		0.8	0.46731	0.23654	0.202500	0.102500
		1.0	0.58333	0.33333	0.233333	0.133333
	1.0	0.0	1.00000	0.62500	0.266667	0.166667
		0.2	1.17814	0.78340	0.298462	0.198462
		0.4	1.43434	1.01768	0.344242	0.244242
		0.6	1.82353	1.38235	0.413333	0.313333
		0.8	2.46429	1.99554	0.525714	0.425714
		1.0	3.66667	3.16667	0.733333	0.633333

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