DETECTING INFEASIBILITY AND FIXING VARIABLES IN 0-1 LINEAR PROGRAMMING PROBLEMS

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RESUMEN

En este trabajo se presenta un procedimiento de obtención de cotas inferiores para una función lineal a partir de ciertas familias de empaquetamientos, cubrimientos y conjuntos ordenados especiales. Asimismo, se presentan nuevos métodos de detección de infactibilidad y fijación de variables en problemas de programación lineal 0-1 basados en dichas cotas que permiten considerar conjuntamente varias restricciones. Además, se muestran algunas situaciones que son detectadas por estos métodos, pero no por los métodos tradicionales, los cuales consideran las restricciones individualmente.

Palabras clave: Infactibilidad, empaquetamientos, cubrimientos, conjuntos ordenados especiales, familias admisibles.

ABSTRACT

In this paper we present a procedure for obtaining lower bounds on a linear function by means of certain families of packings, coverings and special ordered sets. We also present new methods for detecting infeasibility and fixing variables in 0-1 linear programming problems based on these bounds that allow consideration of several constraints jointly. Furthermore, we show some situations which are detected by these new methods, but not by the traditional methods, which consider the constraints individually.

Key words: Infeasibility, packings, coverings, special ordered sets, admissible families.

MSC:90C10

1. INTRODUCTION

Consider the 0-1 linear programming problem

$$\max \left\{ \sum_{j\in J} c_j x_j \mid \sum_{j\in J} a_{ij} x_j \sim b_i \ \ \forall i \in I,\ x_j \in \{0,1\} \ \ \forall j \in J \right\}, \qquad \text{(P)}$$

where $J = \{1,...,n\}$, $I = \{1,...,m\}$, $\{c_j\}_{j\in J}$, $\{a_{ij}\}_{i\in I, j\in J}$, $\{b_i\}_{i\in I}$ are rational numbers and $\sim$ is the sense of each constraint ($\leq$, $\geq$, $=$).

In integer programming there are many ways of representing the same problem, and the choice of the formulation is of crucial importance to solving it [see e.g. Hoffmann-Padberg (1991), Johnson **et al**. (2000), Nemhauser-Wolsey (1988) and Savelsberg (1944)].

Preprocessing attempts to improve the initial formulation by using several automatic techniques such as unfeasibility and redundancy detection, variable fixing and constraint reformulation [see Crowder **et al**. (1983), Escudero-Muñoz (1998), Hoffmann-Padberg (1991), Johnson **et al**. (1985), Muñoz (1999), (2000) and Savelsberg (1994) among others].

It is well known that preprocessing techniques can considerably reduce the time required to solve large-scale integer programming problems.

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The detection of the infeasibility of problem (P) is based on computing bounds on a linear function z whose variables $(x_j)_{j \in J}$ are restricted to take values in a certain subset of $\{0,1\}^n$. Obviously, the best bound is the one given by the optimal value of z in the associated optimization problem. However, in general, this problem is not easy to solve, since it is similar to (P), see Section 4. Hence, there is a need to develop simple procedures for obtaining bounds on z.

In easy terms, a packing, a covering and a special ordered set can be considered as subsets of indices of 0'1 variables where at most, at least and exactly one such variable, respectively, can take the value 1. These structures may appear explicitly in the problem but can also be derived from the constraint system by using probing techniques [see Atantür **et al**.(2000), Guignard-Spielberg (1981), Savalsberg (1994)]. Other methods for packing identification can be found in Dietrich **et al**. (1996), Muñoz (1999). See also Muñoz (1995).

The earliest papers dealing with obtaining bounds on linear functions consider only the coefficients of those functions [see Crowder **et al**. (1983), and Savelsberg (1994) among others]. In 1985 Johnson, Kostreva and Suhl introduced a more advanced procedure that makes use of information from families of pointwise disjoint packings [see Hoffmann-Padberg (1991), Johnson **et al**. (1985)] and, in 1996 Escudero, Garín and Pérez improved this procedure allowing overlapping among certain pairs of packings [see Escudero **et al**. (1996), Muñoz (1999)].

This theoretical paper whose contribution is twofold. First, we extend the procedure given in Escudero **et al**. (1996) to obtain lower bounds on linear functions, using certain families of packings, covering and special ordered sets, so called admissible families. Secondly, we present new methods for detecting unfeasibility and fixing variables in 0-1 linear programming problems that allow consideration of several constraints jointly.

These methods can easily be generalized to mixed programming problems with bounded variables [see e.g. Savelsberg (1994)].

The paper is organized as follows: Section 2 reviews the concepts of packings, coverings and special ordered sets. Section 3 introduces the concept of admissible families, describes a procedure for obtaining lower bounds on a linear function based on this type of families, and provides an example in which the procedures using only families of packings obtain worse lower bounds. Sections 4 and 5 present our methods for detecting infeasibility and fixing variables in problem (P), respectively. They also show some situations detected by these methods, but not by the methods available in current literature, which consider single constraints. Finally, Section 6 draws some conclusions from this work.

2. PACKINGS, COVERINGS AND SPECIAL ORDERED SETS

Given a set of variables $\{x_1,...,x_n\}$ and a set $F \subseteq \{1,...,n\}$, let X(F) denote the sum of the variables whose indices belong to F, that is, $X(F) = \sum_{j \in F} x_j$.

Based on the notation used in [2], we define the following concepts:

Definition 1. A **packing** C is a non-empty subset of indices of 0-1 variables that induces the constraint $X(C^+) - X(C^-) \leq 1 - |C^-|$, where $C^+ \cup C^- = C$ and $C^+ \cap C^- = \emptyset$.

Definition 2. A **covering** C is a non-empty subset of indices of 0 - 1 variables that induces the constraint $X(C^+) - X(C^-) \geq 1 - |C^-|$, where $C^+ \cup C^- = C$ and $C^+ \cap C^- = \emptyset$.

Definition 3. A **special ordered set** C is a non-empty subset of indices of 0-1 variables that induces the constraint $X(C^+) - X(C^-) = 1 - |C^-|$, where $C^+ \cup C^- = C$ and $C^+ \cap C^- = \emptyset$.

Lemma 1 proves that any proper subset of a packing or of a special ordered set is a packing.

Lemma 1. Let C be a packing or a special ordered set, let C' be a proper subset of C and let $(x_j)_{j\in C} \in \{0,1\}^{|C|}$ be a feasible solution for the constraint induced by C. Then $\sum_{j\in C'\cap C^+} x_j - \sum_{j\in C'\cap C^-} x_j \le 1 - |C' \cap C^-|$.

Proof. Since $C'\cap C^+ = C^+ \setminus (C^+ \setminus C')$, $C'\cap C^- = C^- \setminus (C^- \setminus C')$, $\sum_{j\in C^+} x_j - \sum_{j\in C^-} x_j \le 1 - |C^-|$ and $x_j \in \{0,1\}$ $\forall j \in C$, we obtain that $\sum_{j\in C'\cap C^+} x_j - \sum_{j\in C'\cap C^-} x_j = \sum_{j\in C^+} x_j - \sum_{j\in C^-} x_j - \sum_{j\in C^+\setminus C'} x_j + \sum_{j\in C^-\setminus C'} x_j \le 1 - |C^-| + |C^- \setminus C'| = 1 - |C'\cap C^-|$.

3. OBTAINING LOWER BOUNDS ON A LINEAR FUNCTION

Definition 4. A constraint with variables $x_1,...,x_n$ is said to be **valid** for a set $R \subseteq IR^n$ if it is satisfied by any vector $(x_1,...,x_n) \in R$.

Definition 5. Let $\{C_k\}_{k\in K_1}$, $\{C_k\}_{k\in K_2}$ and $\{C_k\}_{k\in K_3}$ be a family of packings, coverings and special ordered sets respectively. The family $\{C_k\}_{k\in K}$, where $K = K_1 \cup K_2 \cup K_3$, is said to be **admissible** for a set $R \subseteq \{0,1\}^n$ if the constraints induced by $\{C_k\}_{k\in K}$ are valid for R and each set K_l with $l \in \{1,2,3\}$ can be expressed as the union of three pairwise disjoint sets, say D_l, S_l and $\overline{S}_l$, that satisfy the following conditions, where $D = D_1 \cup D_2 \cup D_3$, $S = S_1 \cup S_2 \cup S_3$ and $\overline{S} = \overline{S}_1 \cup \overline{S}_2 \cup \overline{S}_3$:

(1) If $k \in D$ and $k' \in K \setminus \{k\}$, then $C_k \cap C_{k'} = \varnothing$.

(2) For each $k \in S$ there exists a unique $\overline{s}(k) \in \overline{S}$ such that
$C_k \cap C_{\overline{s}(k)} = (C_k^+ \cap C_{\overline{s}(k)}^{\pm}) \cup (C_k^- \cap C_{\overline{s}(k)}^{-}) \ne \varnothing$, and $C_k \cap C_{k'} = \varnothing$
$\forall k' \in (S \setminus \{k\}) \cup (\overline{S} \setminus \{\overline{s}(k)\})$.

(3) For each $k \in \overline{S}$ there exists a unique $s(k) \in S$ such that $\overline{s}(s(k)) = k$, and
$C_k \cap C_{k'} = \varnothing$ $\forall k' \in \overline{S} \setminus \{k\}$.

NOTE. By conditions (2) and (3) above, we have that $|S| = |\overline{S}|$.

All of the results stated from now on can be generalized to admissible families such that the constraints induced by $\{C_k\}_{k\in K}$ are of the form $X(C_k^+) - X(C_k^-) \le n_k - |C_k^-|$, $X(C_k^+) - X(C_k^-) \ge n_k - |C_k^-|$ and $X(C_k^+) - X(C_k^-) = n_k - |C_k^-|$, where n_k is an integer with $1 \le n_k \le |C_k|$.

Given a non-empty set $R \subseteq \{0,1\}^n$, we are interested in obtaining lower bounds on a function $z = \sum_{j\in J} a_j x_j$, where $\{a_j\}_{j\in J}$ are rationals and $(x_j)_{j\in J} \in R$. For that, we consider an admissible family for R, say $C = \{C_k\}_{k\in K}$, where $K = K_1 \cup K_2 \cup K_3$ and $\{C_k\}_{k\in K_1}$, $\{C_k\}_{k\in K_2}$ and $\{C_k\}_{k\in K_3}$ are a family of packings, coverings and special ordered sets respectively. Without loss of generality let us assume that $C_k^- = \varnothing$ for each $k \in K$ (otherwise, it suffices to substitute x_j by $1 - x'_j$ $\forall j \in \bigcup_{k\in K} C_k^-$).

Let $l_{z,C} = \min\left\{\sum_{j\in J} a_j x_j \mid (x_j)_{j\in J} \in R_c\right\}$, where $R_c = \{(x_j)_{j\in J} \in \{0,1\}^n \mid X(C_k) \le 1 \ \ \forall k \in K_1,\ X(C_k) \ge 1 \ \ \forall k \in K_2,\ X(C_k) = 1 \ \ \forall k \in K_3\}$ (if $K = \varnothing$, we define $R_c = \{0,1\}^n$). Then $R \subseteq R_c$, since the constraints induced by $\{C_k\}_{k\in K}$ are valid for R. Thus, $l_{z,C}$ is a lower bound on the function z.

Below we give some cases where eliminating one of the elements of C leaves the set R_c unchanged and, so, the bound $l_{z,C}$ also remains unchanged.

Let $k, k' \in S \cup \overline{S}$ be such that $C_k \subset C_{k'}$.

- If $k \in S_1 \cup \overline{S}_1$ and $k' \in S_1 \cup \overline{S}_1 \cup S_3 \cup \overline{S}_3$, we can eliminate k from $S_1 \cup \overline{S}_1$ and move k' to $D_1 \cup D_3$.
- If $k \in S_1 \cup \overline{S}_1$, $|C_k| = 1$ and $k' \in S_2 \cup \overline{S}_2$, we can eliminate k from $S_1 \cup \overline{S}_1$ and move k' to D_2.
- If $k \in S_2 \cup \overline{S}_2 \cup S_3 \cup \overline{S}_3$ and $k' \in S_2 \cup \overline{S}_2$, we can eliminate k' from $S_2 \cup \overline{S}_2$ and move k to $D_2 \cup D_3$.
- If $k \in S_2 \cup \overline{S}_2 \cup S_3 \cup \overline{S}_3$ and $k' \in S_1 \cup \overline{S}_1 \cup S_3 \cup \overline{S}_3$, we can fix $x_j = 0 \ \ \forall j \in C_{k'} \setminus C_k$, eliminate k' from $S_1 \cup \overline{S}_1 \cup S_3 \cup \overline{S}_3$ and move k to D_3.

Now, let $k \in D$ be such that $C_k = \{j\}$.

- If $k \in D_1$, we can eliminate k from D_1.
- If $k \in D_2 \cup D_3$, we can fix $x_j = 1$ and eliminate k from $D_2 \cup D_3$.

Therefore, we can assume that, for each $k \in K$, $|C_k| > 1$ and, if $\exists\ k' \in K \setminus \{k\}$ such that $C_k \subset C_{k'}$, then $k \in S_1 \cup \overline{S}_1$ and $k' \in S_2 \cup \overline{S}_2$.

Any non-empty subset of R_C containing R verifies that, if $(x_j)_{j\in J}$ is restricted to belong to that subset, then $l_{z,c}$ is still a lower bound on z. Consequently, whenever $l_{z,C}$ is mentioned, it will be assumed that $(x_j)_{j\in J}$ can take any value in R_C and the initial set R will be allowed to be empty. (Note that $R_C \neq \varnothing$).

Lemma 2. Let $R_{z,b,C} = \left\{(x_j)_{j\in J} \in R_C \mid \sum_{j\in J} a_j x_j \le b\right\}$, where b is a rational constant.

Then $R_{z,b,C} = \varnothing$ if and only if $l_{z,C} > b$.

Proof. It follows from the definition of $l_{z,C}$.

Given a set $C \subseteq J$, we define $\underline{\varphi}(C) = \min\{a_j \mid j \in C\}$ and $\underline{\varphi}^0(C) = \min\{\underline{\varphi}(C), 0\}$.

Let $J^+ = \{j \in J \mid a_j > 0\}$, $J^- = \{j \in J \mid a_j < 0\}$, $J^0 = \{j \in J \mid a_j = 0\}$, $T = \bigcup_{k\in K} C_k$ and

$$
I_k = \begin{cases}
\underline{\varphi}^0(C_k) & \forall k \in D_1 \\
\underline{\varphi}(C_k) & \forall k \in D_2 \text{ such that } C_k \subseteq J^+ \\
\sum_{j \in C_k \cap J^-} a_j & \forall k \in D_2 \text{ such that } C_k \not\subseteq J^+ \\
\underline{\varphi}(C_k) & \forall k \in D_3 \\
\min\{\underline{\varphi}(C_k \cap C_{\bar{s}(k)}), \underline{\varphi}^0(C_k \setminus C_{\bar{s}(k)}) + \underline{\varphi}^0(C_{\bar{s}(k)} \setminus C_k)\} & \forall k \in S_1 \text{ such that } \bar{s}(k) \in \overline{S}_1 \\
\min\{\underline{\varphi}(C_k \cap C_{\bar{s}(k)}), \underline{\varphi}^0(C_k \setminus C_{\bar{s}(k)}) + \underline{\varphi}(C_{\bar{s}(k)} \setminus C_k)\} & \forall k \in S_1 \text{ such that } \bar{s}(k) \in \overline{S}_2 \text{ and } C_{\bar{s}(k)} \setminus C_k \subseteq J^+ \\
\underline{\varphi}(C_k) + \sum_{j \in (C_{\bar{s}(k)} \setminus C_k) \cap J^-} a_j & \forall k \in S_1 \text{ such that } \bar{s}(k) \in \overline{S}_2 \text{ and } C_{\bar{s}(k)} \setminus C_k \not\subseteq J^+ \\
\min\{\underline{\varphi}(C_k \cap C_{\bar{s}(k)}), \underline{\varphi}^0(C_k \setminus C_{\bar{s}(k)}) + \underline{\varphi}(C_{\bar{s}(k)} \setminus C_k)\} & \forall k \in S_1 \text{ such that } \bar{s}(k) \in \overline{S}_3 \\
\min\{\underline{\varphi}(C_k \cap C_{\bar{s}(k)}), \underline{\varphi}(C_k \setminus C_{\bar{s}(k)}) + \underline{\varphi}^0(C_{\bar{s}(k)} \setminus C_k)\} & \forall k \in S_2 \text{ such that } \bar{s}(k) \in \overline{S}_1 \text{ and } C_k \setminus C_{\bar{s}(k)} \subseteq J^+ \\
\sum_{j \in (C_k \setminus C_{\bar{s}(k)}) \cap J^-} a_j + \underline{\varphi}^0(C_{\bar{s}(k)}) & \forall k \in S_2 \text{ such that } \bar{s}(k) \in \overline{S}_1 \text{ and } C_k \setminus C_{\bar{s}(k)} \not\subseteq J^+ \\
\min\{\underline{\varphi}(C_k \cap C_{\bar{s}(k)}), \underline{\varphi}(C_k \setminus C_{\bar{s}(k)}) + \underline{\varphi}(C_{\bar{s}(k)} \setminus C_k)\} & \forall k \in S_2 \text{ such that } \bar{s}(k) \in \overline{S}_2 \text{ and } C_k \cup C_{\bar{s}(k)} \subseteq J^+ \\
\underline{\varphi}(C_k) + \sum_{j \in (C_{\bar{s}(k)} \setminus C_k) \cap J^-} a_j & \forall k \in S_2 \text{ such that } \bar{s}(k) \in \overline{S}_2,\ C_k \subseteq J^+ \text{ and } C_{\bar{s}(k)} \setminus C_k \subseteq J^+ \\
\sum_{j \in (C_k \setminus C_{\bar{s}(k)}) \cap J^-} a_j + \underline{\varphi}(C_{\bar{s}(k)}) & \forall k \in S_2 \text{ such that } \bar{s}(k) \in \overline{S}_2,\ C_k \not\subseteq J^+ \text{ and } C_{\bar{s}(k)} \not\subseteq J^+ \\
\sum_{j \in (C_k \cup C_{\bar{s}(k)}) \cap J^-} a_j & \forall k \in S_2 \text{ such that } \bar{s}(k) \in \overline{S}_2,\ C_k \not\subseteq J^+ \text{ and } C_{\bar{s}(k)} \not\subseteq J^+ \\
\min\{\underline{\varphi}(C_k \cap C_{\bar{s}(k)}), \underline{\varphi}(C_k \setminus C_{\bar{s}(k)}) + \underline{\varphi}(C_{\bar{s}(k)} \setminus C_k)\} & \forall k \in S_2 \text{ such that } \bar{s}(k) \in \overline{S}_3 \text{ and } C_k \setminus C_{\bar{s}(k)} \subseteq J^+ \\
\sum_{j \in (C_k \setminus C_{\bar{s}(k)}) \cap J^-} a_j + \underline{\varphi}(C_{\bar{s}(k)}) & \forall k \in S_2 \text{ such that } \bar{s}(k) \in \overline{S}_3 \text{ and } C_k \setminus C_{\bar{s}(k)} \not\subseteq J^+ \\
\min\{\underline{\varphi}(C_k \cap C_{\bar{s}(k)}), \underline{\varphi}(C_k \setminus C_{\bar{s}(k)}) + \underline{\varphi}^0(C_{\bar{s}(k)} \setminus C_k)\} & \forall k \in S_3 \text{ such that } \bar{s}(k) \in \overline{S}_1 \\
\min\{\underline{\varphi}(C_k \cap C_{\bar{s}(k)}), \underline{\varphi}(C_k \setminus C_{\bar{s}(k)}) + \underline{\varphi}(C_{\bar{s}(k)} \setminus C_k)\} & \forall k \in S_3 \text{ such that } \bar{s}(k) \in \overline{S}_2 \text{ and } C_{\bar{s}(k)} \setminus C_k \subseteq J^+ \\
\underline{\varphi}(C_k) + \sum_{j \in (C_{\bar{s}(k)} \setminus C_k) \cap J^-} a_j & \forall k \in S_3 \text{ such that } \bar{s}(k) \in \overline{S}_2 \text{ and } C_{\bar{s}(k)} \setminus C_k \not\subseteq J^+ \\
\min\{\underline{\varphi}(C_k \cap C_{\bar{s}(k)}), \underline{\varphi}(C_k \setminus C_{\bar{s}(k)}) + \underline{\varphi}(C_{\bar{s}(k)} \setminus C_k)\} & \forall k \in S_3 \text{ such that } \bar{s}(k) \in \overline{S}_3
\end{cases}
$$

Theorem 1. $l_{z,C} = \sum_{k\in D\cup S} l_k + \sum_{j\in J^- \setminus T} a_j$.

PROOF. Since the sets $\{C_k\}_{k\in D}, \{C_k \cup C_{\bar{s}(k)}\}_{k\in S}$ are pairwise disjoint, it can easily be verified that $\sum_{j\in J} a_j x_j \geq \sum_{k\in D\cup S} l_k + \sum_{j\in J^-\setminus T} a_j \quad \forall (x_j)_{j\in J} \in R_c$. On the other hand, it is clear that $\exists (x_j^*)_{j\in J} \in R_c$ such that $\sum_{j\in C_k} a_j x_j^* = l_k \quad \forall k \in D$, $\sum_{j\in C_k \cup C_{\bar{s}(k)}} a_j x_j^* = l_k \quad \forall k \in S$ and $\sum_{j\in J\setminus T} a_j x_j^* = \sum_{j\in J^-\setminus T} a_j$. Hence, we have that $\sum_{j\in J} a_j x_j^* = \sum_{k\in D\cup S} l_k + \sum_{j\in J^-\setminus T} a_j$, which proves the assertion.

Corollary 1. Let z' = λz, where λ is a non-negative rational constant. Then $l_{z',C}$.

Corollary 2. $l_{z,C} \leq 0$ if $K_2 = K_3 = \varnothing$.

Corollary 3. $l_{z,C} = \min\left\{ \sum_{j\in J} a_j x_j \middle| (x_j)_{j\in J} \in [0,1]^n, X(C_k) \leq 1\, \forall k \in K_1, X(C_k) \geq 1\, \forall k \in K_2, X(C_k) = 1\, \forall k \in K_3 \right\}$.

In general, there will exist several admissible families for the set R. Example 1 illustrates the great variations in the value that $l_{z,C}$ takes depending on the family C that has been selected; this demonstrates the importance of making a good choice.

Example 1. Let $z = -x_1 + 4x_2 - 2x_3 - 6x_4 + x_5 + 3x_6 + 5x_7 - 4x_8$ and let R be the set of solutions $(x_1,...,x_8) \in \{0,1\}^8$ that satisfy the following constraints:

$$x_1 + x_2 + x_3 + x_7 \leq 1 \quad (1)$$

$$x_3 + x_4 + x_7 \leq 1 \quad (2)$$

$$x_2 + x_6 + x_7 \geq 1 \quad (3)$$

$$x_4 + x_5 + x_8 = 1 \quad (4)$$

Consider the packings $C_1 = \{1,2,3,7\}$ and $C_2 = \{3,4,7\}$, the covering $C_3 = \{2,6,7\}$ and the special ordered set $C_4 = \{4,5,8\}$. (Note that constraints (1)-(4) are induced by $C_1,...,C_4$ respectively).

The family $\{C_1,C_2,C_3,C_4\}$ is not admissible for R, since $C_1 \cap C_2 \cap C_3 \neq \varnothing$. Nevertheless, by Lemma 1 any non-empty set $C \subset C_k$, where $k \in \{1,2,4\}$, is a packing whose induced constraint $X(C) \leq 1$ is valid for R.

Let $C' = \{C'_1, C'_2, C'_3\}$, where $C'_1 = \{1,2,3,7\}, C'_2 = \{3,4,7\}$ and $C'_3 = \{5,8\}$. Taking $D_1 = \{3\}$, $S_1 = \{1\}$, $\bar{S}_1 = \{2\}$ and $D_2 = S_2 = \bar{S}_2 = D_3 = S_3 = \bar{S}_3 = \varnothing$ we have that C' is an admissible family for R and, by Theorem 1, $l_{z,C'} = l_1 + l_3 = \min\{-2, -1 -6\} -4 = -11$.

Let $C'' = \{C''_1, C''_2, C''_3\}$, where $C''_1 = \{3,4,7\}$, $C''_2 = \{2,6,7\}$ and $C''_3 = \{5,8\}$. Taking $D_1 = \{3\}$, $S_1 = \{1\}$, $\bar{S}_2 = \{2\}$ and $\bar{S}_1 = D_2 = S_2 = D_3 = S_3 = \bar{S}_3 = \varnothing$ we have that C'' is an admissible family for R and, by Theorem 1, $l_{z,C''} = l_1 + l_3 + a_1 = \min\{5, -6c + 3\} -4 -1 = -8$.

Let $C''' = \{C'''_1, C'''_2, C'''_3, C'''_4\}$, where $C'''_1 = \{1,2,7\}$, $C'''_2 = \{3,4\}$, $C'''_3 = \{2,6,7\}$ and $C'''_4 = \{4,5,8\}$. Taking $S_1 = \{1,2\}$, $\bar{S}_2 = \{3\}$, $\bar{S}_3 = \{4\}$ and $D_1 = \bar{S}_1 = D_2 = S_2 = D_3 = S_3 = \varnothing$ we have that C''' is an admissible family for R and, by Theorem 1, $l_{z,c'''} = l_1 + l_2 = \min\{4, -1+3\} + \min\{-6, -2 -4\} = -4$.

The best lower bound on the function z is given by C''', since $l_{z,C'} < l_{z,C''} < l_{z,C'''}$. Furthermore, choosing $x_1 = x_4 = x_6 = 1$ and $x_2 = x_3 = x_5 = x_7 = x_8 = 0$, we obtain that $(x_1,...,x_8) \in R$ and $-x_1 + 4x_2 - 2x_3 - 6x_4 + x_5 + 3x_6 + 5x_7 - 4x_8 = -4$. Consequently, there is no lower bound on z stronger than $l_{z,C'''}$. (Note that if one restricts the admissible families for R to families of packings, as the traditional procedures do, then the associated lower bounds on z will be less than –4, since $\min\{-x_1 + 4x_2 - 2x_3 - 6x_4 + x_5 + 3x_6 + 5x_7 - 4x_8 \mid \sum_{j\in C_k} x_j \le 1$ $\forall k \in \{1,2,4\},\ x_j \in \{0,1\}\ \ \forall j \in \{1,...,8\}\} = -7$).

4. DETECTING INFEASIBILITY

Let $R = \{(x_j)_{j\in J} \in \{0,1\}^n \mid \sum_{j\in J} a_{ij}x_j \sim b_i\ \forall i \in I\}$. We are interested in determining whether (P) is an infeasible problem, that is, whether $R = \varnothing$.

Let $C = \{C_k\}_{k\in K}$ be an admissible family for R, where $K = K_1 \cup K_2 \cup K_3$ and $\{C_k\}_{k\in K_1}$, $\{C_k\}_{k\in K_2}$ and $\{C_k\}_{k\in K_3}$ are a family of packings, coverings and special ordered sets respectively. The family C will be obtained as follows:

We start by identifying a family C_0 of packings, coverings and special ordered sets whose induced constraints are valid for R and that contains the family C_p of packings, coverings and special ordered sets that induce contraints of problem (P), see Section 1.

If C_0 is an admissible family for R, we take $C = C_0$; otherwise, by Lemma 1 it is easy to determine an admissible family for R from C_0, see Example 1. For simplicity, we assume that, for each $k \in K$, $C_k^- = \varnothing$, $|C_k| > 1$ and, if $\exists\ k' \in K \setminus \{k\}$ such that $C_k \subset C_{k'}$, then $k \in S_1 \cup \overline{S}_1$ and $k' \in S_2 \cup \overline{S}_2$.

Let I_C be the set of indices of the constraints in (P) that are induced by $C_k\}_{k\in K}$.

Without loss of generality, from now on we assume that every constraint in (P) is an inequality of type ≤. (Note that any inequality of type ≥ can be converted into another one of type ≤ by multiplying it by –1, and any equality can be decomposed into two inequalities).

Proposition 1. Let $\sum_{j\in J} a_j x_j \le b$ be a valid inequality for R and let $z = \sum_{j\in J} a_j x_j$. If $l_{z,C} > b$, then (P) is an infeasible problem.

PROOF. If $l_{z,C} > b$, by Lemma 2 it follows that $R_{z,b,C} = \varnothing$ and, since $R \subseteq R_{z,b,C}$, we have that (P) is an infeasible problem.

Proposition 2. Let $i_1,...,i_p \in I \setminus I_C$ be such that $i_l \ne i_{l'}$ $\forall l, l' \in \{1,...,p\}$ with $l \ne l'$, and let $z = \sum_{j\in J} (\lambda_1 a_{i_1 j} + ... + \lambda_p a_{i_p j}) x_j$ where $p \ge 1$ and $\lambda_1,...,\lambda_p$ are positive integers relatively prime. If $l_{z,C} > \lambda_1 b_{i_1} + ... + \lambda_p b_{i_p}$, then (P) is an infeasible problem.

PROOF. It follows from Proposition 1.

In Proposition 2 it is not necessary to impose the condition that $\lambda_1,...,\lambda_p$ be integers relatively prime. Assuming they are integers, $\lambda_1 b_{i_1} + ... + \lambda_p b_{i_p}$ and the coefficients of the function z are rationals. On the other hand, if $\lambda_1,...,\lambda_p$ are positive integers, it follows from Corollary 1 that the result of applying Proposition 2 by considering $\lambda_1,...,\lambda_p$ is the same as by considering $\frac{\lambda_1}{M},\dots,\frac{\lambda_p}{M}$ where M is the greatest common divisor of $\lambda_1,..., \lambda_p$. Therefore, in order to make the calculation of $\lambda_1 b_{i_1} + ... + \lambda_p b_{i_p}$ and $l_{z,C}$ easier, it is advisable that $\lambda_1,..., \lambda_p$ be relatively prime (if p = 1, we will take $\lambda_1 = 1$).

It is clear that Proposition 2 also holds for any indices $i_1,..., i_p \in I$. Now, if $i_1,..., i_p \in I_C$, taking $b = \lambda_1 b_{i_1} +...+ \lambda_p b_{i_p}$ in Lemma 2 we have that $R_{z,b,C} = R_C \neq \emptyset$, hence $l_{z,C} \leq \lambda_1 b_{i_1} +...+ \lambda_p b_{i_p}$ and Proposition 2 does not detect the infeasibility of problem (P). If not all indices $i_1,...,i_p$ belong to I_C, Lemma 3 proves that if Proposition 2 detects the infeasibility of (P) by considering $i_1,...,i_p$, then it will also detect it by considering only those indices in $I \setminus I_C$.

Lemma 3. Let $i_1,...,i_{p'} \in I \setminus I_C$ be such that $i_l \neq i_{l'}$ $\forall l,l' \in \{1,...p'\}$ with $l \neq l'$, let $i_{p'+1},...,i_p \in I_C$ be such that $i_l \neq i_{l'}$ $\forall l,l' \in \{p'+1,...,p\}$ with $l \neq l'$, let $z = \sum_{j\in J}(\lambda_1 a_{i_1 j} + ... + \lambda_p a_{i_p j})x_j$ and $z' = \sum_{j\in J}(\lambda_1 a_{i_1 j} + ... + \lambda_{p'} a_{i_{p'} j})x_j$, where $1 \leq p' < p$ and $\lambda_1,...,\lambda_p$ are positive integers relatively prime. If $l_{z,C} > \lambda_1 b_{i_1} +...+ \lambda_p b_{i_p}$, then $l_{z',C} > \lambda_1 b_{i_1} +...+ \lambda_{p'} b_{i_{p'}}$.

PROOF. Let $b = \lambda_1 b_{i_1} +...+ \lambda_p b_{i_p}$ and $b' = \lambda_1 b_{i_1} +...+ \lambda_{p'} b_{i_{p'}}$. If $l_{z,C} > b$, by Lemma 2 we have that $R_{z,b,C} = \emptyset$ and, since $i_{p'+1},...,i_p \in I_C$ and $\lambda_{p'+1},..., \lambda_p > 0$, it follows that $R_{z',b',C} \subseteq R_{z,b,C}$, hence $R_{z',b',C} = \emptyset$ and $l_{z',c} > b'$.

Given $i_1,..., i_p \in I \setminus I_C$, if $K_2 = K_3 = \emptyset$ and $\lambda_1 b_{i_1} +...+ \lambda_p b_{i_p} \geq 0$ $\forall \lambda_1,..., \lambda_p > 0$, it is not necessary to apply Proposition 2, since $l_{z,C} \leq 0$ by Corollary 2 and, so, $l_{z,C} \leq \lambda_1 b_{i_1} +...+ \lambda_p b_{i_p}$.

Example 2. Let (5)-(7) be the constraint system that defines the feasible region of (P).

$$-2x_1 \qquad - x_3 \leq -2 \tag{5}$$

$$-2x_2 - x_3 \leq -2 \tag{6}$$

$$x_1 + x_2 \qquad \leq 1 \tag{7}$$

By constraints (5) and (6), we have that $x_1 = x_2 = 1$ in every feasible solution to (P).

Consequently, by constraint (7) it follows that (P) is an infeasible problem.

Let $C = \{\{1, 2\}\}$. (Note that constraint (7) is induced by the packing $\{1,2\}$). Taking $x_1 = x_2 = \frac{1}{2}$ and $x_3 = 1$ we obtain a solution in $[0,1]^3$ that satisfies constraints (5) - (7).

Thus, by Corollary 3 we can conclude that the infeasibility of problem (P) will not be detected by applying Proposition 2. Nevertheless, it can be detected by using some of the results stated in Section 5, see Example 4.

Example 3 shows an infeasibility situation which is detected by considering two constraints jointly, but not by considering them individually.

Example 3. Let (8)-(11) be the constraint system that defines the feasible region of (P).

$$-2x_1 \qquad - x_3 \qquad \leq -2 \tag{8}$$

$$-2x_2 \qquad - x_4 \leq -1 \tag{9}$$

$$x_1 + x_2 + x_3 \qquad \leq 1 \tag{10}$$

$$x_1 \qquad + x_4 \leq 1 \tag{11}$$

Let $C = \{\{1, 2, 3\}, \{1,4\}\}$. (Note that constraints (10) and (11) are induced by the packings $\{1, 2, 3\}$ and $\{1, 4\}$ respectively). By applying Proposition 2 to constraints (8) and (9) we have that $z = -2\lambda_1 x_1 - 2\lambda_2 x_2 - \lambda_1 x_3 - \lambda_2 x_4$, $l_{z,C} = \min\{-2\lambda_1, \min\{-2\lambda_2, -\lambda_1\} - \lambda_2\}$ and $\lambda_1 b_{i_1} + \lambda_2 b_{i_2} = -2\lambda_1 - \lambda_2$.

- If $\lambda_1 \le \lambda_2$, then $l_{z,C} = -3\lambda_2 = -2\lambda_1 - \lambda_2 + 2\lambda_1 - 2\lambda_2 \le \lambda_1 b_{i_1} + \lambda_2 b_{i_2}$.

- If $\lambda_2 < \lambda_1 \le \frac{3}{2}\lambda_2$, then $l_{z,C} = -3\lambda_2 = -2\lambda_1 - \lambda_2 + 2\lambda_1 - 2\lambda_2 > \lambda_1 b_{i_1} + \lambda_2 b_{i_2}$.

- If $\lambda_1 > \frac{3}{2}\lambda_2$, then $l_{z,C} = -2\lambda_1 > \lambda_1 b_{i_1} + \lambda_2 b_{i_2}$.

So, to detect the infeasibility of problem (P) it suffices to choose λ_1 and λ_2 such that $\lambda_1 > \lambda_2$. However, if Proposition 2 is applied separately to constraints (8) and (9), the infeasibility is not detected, since taking $\lambda_1 = 1$ we obtain that $l_{z,C} = -2 = b_{i_1}$ and $l_{z,C} = -3 < b_{i_1}$ respectively.

5. FIXING VARIABLES

In this section we describe a methodology for fixing variables using feasibility testing, see Section 4. Other methodologies for variable fixing can be found in [3, 8, 9, 12].

Let $R = \{(x_j)\}_{j\in J} \in \{0,1\}^n \mid \sum_{j\in J} a_{ij}x_j \le b_i \ \ \forall i \in I\}$ and let $C = \{C_k\}_{k\in K}$ be an admissible family for R, where, for simplicity, we assume that $\{C_k\}_{k\in K}$ is a family of packings. (All of the results stated in this section can be generalized to any admissible family for R). We also assume that, for each $k \in K$, $C_{\bar{k}} = \varnothing$, $|C_k| > 1$ and $\nexists\, k' \in K \setminus \{k\}$ such that $C_k \subset C_{k'}$.

Let I_C be the set of indices of the constraints in (P) that are induced by $\{C_k\}_{k\in K}$.

Proposition 3. Let $i_1,\ldots,i_p \in I \setminus I_c$ be such that $i_l \ne i_{l'}$ $\forall l, l' \in \{1,\ldots,p\}$ with $l \ne l'$, and let $z = \sum_{j\in J}(\lambda_1 a_{i_1 j} + \ldots + \lambda_p a_{i_p j})x_j$, where $p \ge 1$ and $\lambda_1,\ldots,\lambda_p$ are positive integers relatively prime. Then

(1) Let $k \in D$ and $j^* \in C_k$. If $\lambda_1 a_{i_1 j^*} + \ldots + \lambda_p a_{i_p j^*} > \lambda_1 b_{i_1} + \ldots + \lambda_p b_{i_p} - l_{z,C} + l_k$, then $x_{j^*} = 0$ in every feasible solution to (P).

(2) Let $k \in S$ and $j^* \in C_k \cap C_{\bar{s}(k)}$. If $\lambda_1 a_{i_1 j^*} + \ldots + \lambda_p a_{i_p j^*} > \lambda_1 b_{i_1} + \ldots + \lambda_p b_{i_p} - l_{z,C} + l_k$, then $x_{j^*} = 0$ in every feasible solution to (P).

(3) Let $k \in S$ and $j^* \in C_k \setminus C_{\bar{s}(k)}$. If $\lambda_1 a_{i_1 j^*} + \ldots + \lambda_p a_{i_p j^*} > \lambda_1 b_{i_1} + \ldots + \lambda_p b_{i_p} - l_{z,C} + l_k - \underline{\varphi}^0(C_{\bar{s}(k)} \setminus C_k)$, then $x_{j^*} = 0$ in every feasible solution to (P).

(4) Let $k \in S$ and $j^* \in C_{\bar{s}(k)} \setminus C_k$. If $\lambda_1 a_{i_1 j^*} + \ldots + \lambda_p a_{i_p j^*} > \lambda_1 b_{i_1} + \ldots + \lambda_p b_{i_p} - l_{z,C} + l_k - \underline{\varphi}^0(C_k \setminus C_{\bar{s}(k)})$, then $x_{j^*} = 0$ in every feasible solution to (P).

(5) Let $j^* \in J^+ \setminus T$. If $\lambda_1 a_{i_1 j^*} + \ldots + \lambda_p a_{i_p j^*} > \lambda_1 b_{i_1} + \ldots + \lambda_p b_{i_p} - l_{z,C}$, then $x_{j^*} = 0$ in every feasible solution to (P).

PROOF. It suffices to fix $x_{j^*} = 1$ in (P) and apply Proposition 2 to check that the resulting problem is infeasible.

Proposition 4. Let $i_1,\ldots, i_p \in I \setminus I_C$ be such that $i_l \ne i_{l'}$ $\forall l,l' \in \{1,\ldots, p\}$ with $l \ne l'$, and let $z = \sum_{j\in J}(\lambda_1 a_{i_1 j} + \ldots + \lambda_p a_{i_p j})x_j$, where $p \ge 1$ and $\lambda_1,\ldots, \lambda_p$ are positive integers relatively prime. Then

(1) Let $k \in D$ and $j^* \in C_k$. If $l_{k,j^*} > \lambda_1 b_{i_1} + ... + \lambda_p b_{i_p} - l_{z,C} + l_k$, where $l_{k,j^*} = \underline{\varphi}^0(C_k \setminus \{j^*\})$, then $x_{j^*} = 1$ and $x_j = 0 \ \forall j \in C_k \setminus \{j^*\}$ in every feasible solution to (P).

(2) Let $k \in S$ and $j^* \in C_k \cap C_{\bar{s}(k)}$.
If $l_{k,j^*} > \lambda_1 b_{i_1} + ... + \lambda_p b_{i_p} - l_{z,C} + l_k$, where $l_{k,j^*} = \min \{\underline{\varphi}((C_k \cap C_{\bar{s}(k)}) \setminus \{j^*\})$, $\underline{\varphi}^0(C_k \setminus C_{\bar{s}(k)}) + \underline{\varphi}^0(C_{\bar{s}(k)} \setminus C_k)\}$, then $x_{j^*} = 1$ and $x_j = 0 \ \forall j \in (C_k \cup C_{\bar{s}(k)}) \setminus \{j^*\}$ in every feasible solution to (P).

(3) Let $k \in S$ and $j^* \in C_k \setminus C_{\bar{s}(k)}$.
If $l_{k,j^*} > \lambda_1 b_{i_1} + ... + \lambda_p b_{i_p} - l_{z,C} + l_k$, where $l_{k,j^*} = \min\{\underline{\varphi}(C_k \cap C_{\bar{s}(k)})$, $\underline{\varphi}^0((C_k \setminus C_{\bar{s}(k)}) \setminus \{j^*\}) + \underline{\varphi}^0(C_{\bar{s}(k)} \setminus C_k)\}$, then $x_{j^*} = 1$ and $x_j = 0 \ \forall j \in C_k \setminus \{j^*\}$ in every feasible solution to (P).

(4) Let $k \in S$ and $j^* \in C_{\bar{s}(k)} \setminus C_k$.
If $l_{k,j^*} > \lambda_1 b_{i_1} + ... + \lambda_p b_{i_p} - l_{z,C} + l_k$, where $l_{k,j^*} = \min\{\underline{\varphi}(C_k \cap C_{\bar{s}(k)})$, $\underline{\varphi}^0(C_k \setminus C_{\bar{s}(k)}) + \underline{\varphi}^0((C_{\bar{s}(k)} \setminus C_k) \setminus \{j^*\})\}$, then $x_{j^*} = 1$ and $x_j = 0 \ \forall j \in C_{\bar{s}(k)} \setminus \{j^*\}$ in every feasible solution to (P).

(5) Let $j^* \in J^- \setminus T$. If $\lambda_1 a_{i_1 j^*} + ... + \lambda_p a_{i_p j^*} < l_{z,C} - \lambda_1 b_{i_1} - ... - \lambda_p b_{i_p}$, then $x_{j^*} = 1$ in every feasible solution to (P).

PROOF. It suffices to fix $x_{j^*} = 0$ in (P) and apply Proposition 2 to check that the resulting problem is infeasible.

Since Proposition 2 holds for any indices $i_1,...,i_p \in I$, Propositions 3 and 4 also hold. If $i_1,...,i_p \in I_C$, no variable can be fixed by applying Propositions 3 and 4, since Proposition 2 does not detect the infeasibility of the problems considered in the proofs of Propositions 3 and 4. If not all indices $i_1,..., i_p$ belong to I_c, by Lemma 3 it follows that if Propositions 3 or 4 can fix a variable by considering $i_1,...,i_p$, then they will also fix it by considering only those indices in $I \setminus I_C$.

Lemmas 4 and 5 state some necessary conditions for the hypotheses required in the four first claims of Propositions 3 and 4, respectively.

Lemma 4. Let $i_1,..., i_p \in I \setminus I_C$ be such that $i_l \neq i_{l'} \ \forall l, l' \in \{1,..., p\}$ with $l \neq l'$, and let $z = \sum_{j \in J} (\lambda_1 a_{i_1 j} + ... + \lambda_p a_{i_p j}) x_j$, where $p \geq 1$ and $\lambda_1,..., \lambda_p$ are positive integers relatively prime. If $l_{z,C} \leq \lambda_1 b_{i_1} + ... + \lambda_p b_{i_p}$, then

(1) Let $k \in D$ and $j^* \in C_k$. If $\lambda_1 a_{i_1 j^*} + ... + \lambda_p a_{i_p j^*} > \lambda_1 b_{i_1} + ... + \lambda_p b_{i_p} - l_{z,C} + l_k$, then $\lambda_1 a_{i_1 j^*} + ... + \lambda_p a_{i_p j^*} > l_k$.

(2) Let $k \in S$ and $j^* \in C_k \cap C_{\bar{s}(k)}$.
If $\lambda_1 a_{i_1 j^*} + ... + \lambda_p a_{i_p j^*} > \lambda_1 b_{i_1} + ... + \lambda_p b_{i_p} - l_{z,C} + l_k$, then $\lambda_1 a_{i_1 j^*} + ... + \lambda_p a_{i_p j^*} > l_k$.

(3) Let $k \in S$ and $j^* \in C_k \setminus C_{\bar{s}(k)}$. If $\lambda_1 a_{i_1 j^*} + ... + \lambda_p a_{i_p j^*} > \lambda_1 b_{i_1} + ... + \lambda_p b_{i_p} - l_{z,C} + l_k - \underline{\varphi}^0(C_{\bar{s}(k)} \setminus C_k)$, then $\lambda_1 a_{i_1 j^*} + ... + \lambda_p a_{i_p j^*} > l_k - \underline{\varphi}^0(C_{\bar{s}(k)} \setminus C_k)$.

(4) Let $k \in S$ and $j^* \in C_{\bar{s}(k)} \setminus C_k$. If $\lambda_1 a_{i_1 j^*} + ... + \lambda_p a_{i_p j^*} > \lambda_1 b_{i_1} + ... + \lambda_p b_{i_p} - l_{z,C} + l_k - \underline{\varphi}^0(C_k \setminus C_{\bar{s}(k)})$, then $\lambda_1 a_{i_1 j^*} + ... + \lambda_p a_{i_p j^*} > l_k - \underline{\varphi}^0(C_k \setminus C_{\bar{s}(k)})$.

PROOF. Trivial.

Lemma 5. Let $i_1,\dots, i_p \in I \setminus I_C$ be such that $i_l \neq i_{l'}$ $\forall l, l' \in \{1,\dots, p\}$ with $l \neq l'$, and let $z = \sum_{j\in J}(\lambda_1 a_{i_1 j} + \dots + \lambda_p a_{i_p j})x_j$, where $p \geq 1$ and $\lambda_1,\dots, \lambda_p$ are positive integers relatively prime. If $l_{z,c} \leq \lambda_1 b_{i_1} +\dots+ \lambda_p b_{i_p}$, then

(1) Let $k \in D$ and $j^* \in C_k$. If $l_{k,j^*} > \lambda_1 b_{i_1} +\dots+ \lambda_p b_{i_p} - l_{z,C} + l_k$, where

$l_{k,j^*} = \underline{\varphi}^0(C_k \setminus \{j^*\})$, then $l_k < l_{z,C} - \lambda_1 b_{i_1} - \dots - \lambda_p b_{i_p}$, and j^* is the unique index in C_k such that $\lambda_1 a_{i_1 j^*} +\dots+ \lambda_p a_{i_p j^*} = l_k$.

(2) Let $k \in S$ and $j^* \in C_k \cap C_{\bar{s}(k)}$.

If $l_{k,j^*} > \lambda_1 b_{i_1} +\dots+ \lambda_p b_{i_p} - l_{z,C} + l_k$, where $l_{k,j^*} = \min\{\underline{\varphi}((C_k \cap C_{\bar{s}(k)})\setminus \{j^*\}), \underline{\varphi}^0(C_k \setminus C_{\bar{s}(k)}) + \underline{\varphi}^0(C_{\bar{s}(k)} \setminus C_k)\}$, then $l_k < \min\{l_{z,C} - \lambda_1 b_{i_1} - \dots - \lambda_p b_{i_p}, \underline{\varphi}^0(C_k \setminus C_{\bar{s}(k)}) + \underline{\varphi}^0(C_{\bar{s}(k)} \setminus C_k)\}$, and j^* is the unique index in $C_k \cap C_{\bar{s}(k)}$ such that $\lambda_1 a_{i_1 j^*} +\dots+ \lambda_p a_{i_p j^*} = l_k$.

(3) Let $k \in S$ and $j^* \in C_k \setminus C_{\bar{s}(k)}$.

If $l_{k,j^*} > \lambda_1 b_{i_1} +\dots+ \lambda_p b_{i_p} - l_{z,C} + l_k$, where $l_{k,j^*} = \min\{\underline{\varphi}(C_k \cap C_{\bar{s}(k)}),$ $\underline{\varphi}^0((C_k \setminus C_{\bar{s}(k)}) \setminus \{j^*\}) + \underline{\varphi}^0(C_{\bar{s}(k)} \setminus C_k)\}$, then $l_k < \min\{l_{z,C} - \lambda_1 b_{i_1} - \dots - \lambda_p b_{i_p}, \underline{\varphi}(C_k \cap C_{\bar{s}(k)})\}$, $\underline{\varphi}(C_k \setminus C_{\bar{s}(k)}) < 0$ and j^* is the unique index in $C_k \setminus C_{\bar{s}(k)}$ such that $\lambda_1 a_{i_1 j^*} +\dots+ \lambda_p a_{i_p j^*} = \underline{\varphi}(C_k \setminus C_{\bar{s}(k)})$.

(4) Let $k \in S$ and $j^* \in C_{\bar{s}(k)} \setminus C_k$.

If $l_{k,j^*} > \lambda_1 b_{i_1} +\dots+ \lambda_p b_{i_p} - l_{z,C} + l_k$, where $l_{k,j^*} = \min\{\underline{\varphi}(C_k \cap C_{\bar{s}(k)}),$ $\underline{\varphi}^0(C_k \setminus C_{\bar{s}(k)}) + \underline{\varphi}^0((C_{\bar{s}(k)} \setminus C_k) \setminus \{j^*\})\}$, then $l_k < \min\{l_{z,C} - \lambda_1 b_{i_1} - \dots - \lambda_p b_{i_p}, \underline{\varphi}(C_k \cap C_{\bar{s}(k)})\}$, $\underline{\varphi}(C_{\bar{s}(k)} \setminus C_k) < 0$ and j^* is the unique index in $C_{\bar{s}(k)} \setminus C_K$ such that $\lambda_1 a_{i_1 j^*} +\dots+ \lambda_p a_{i_p j^*} = \underline{\varphi}(C_{\bar{s}(k)} \setminus C_k)$.

PROOF. It suffices to note that, if $l_{k,j^*} > \lambda_1 b_{i_1} +\dots+ \lambda_p b_{i_p} - l_{z,C} + l_k$, then $l_k < l_{z,C} - \lambda_1 b_{i_1} - \dots - \lambda_p b_{i_p}$ and $l_{k,j^*} > l_k$, since $l_{k,j^*} \leq 0$ and $\lambda_1 b_{i_1} +\dots+ \lambda_p b_{i_p} - l_{z,C} \geq 0$.

Example 2 showed an infeasibility situation which was not detected by applying Proposition 2. Example 4 shows that this situation will be detected if Proposition 4 is applied previously.

Example 4. Let (12)-(14) be the constraint system that defines the feasible region of (P).

$$-2x_1 - x_3 \leq -2 \tag{12}$$

$$-2x_2 - x_3 \leq -2 \tag{13}$$

$$x_1 + x_2 \leq 1 \tag{14}$$

Let $C = \{C_1\}$, where $C_1 = \{1, 2\}$. By applying claim (1) of Proposition 4 to constraint (12) it follows that $x_1 = 1$ and $x_2 = 0$ in every feasible solution to (P), since taking $\lambda_1 = 1$ we obtain that $z = -2x_1 - x_3$, $l_1 = -2$, $l_{z,C} = -3$ and $l_{1,1} = 0$. Accordingly, fixing the variables x_1 and x_2, the constraint system (12) - (14) reduces to the constraint system (15)-(16).

$$-x_3 \leq 0 \tag{15}$$

$$-x_3 \leq -2 \tag{16}$$

Let $C = \emptyset$. By applying Proposition 2 to constraint (16) it follows that (P) is an infeasible problem.

Example 5 shows a situation where all of the variables of a problem can be fixed by considering two constraints jointly, but none of them can be fixed by considering the constraints individually.

Example 5. Let (17)-(21) be the constraint system that defines the feasible region of (P).

$$-4x_1 + 2x_2 - 2x_3 + 3x_4 - 2x_5 + x_6 \leq -2 \quad (17)$$

$$4x_1 - 6x_2 + 3x_3 - 4x_4 + 3x_5 - 3x_6 \leq 0 \quad (18)$$

$$x_1 + x_5 \leq 1 \quad (19)$$

$$x_2 + x_4 \leq 1 \quad (20)$$

$$x_3 + x_5 + x_6 \leq 1 \quad (21)$$

Let $C = \{C_1, C_2, C_3\}$, where $C_1 = \{1, 5\}$, $C_2 = \{2, 4\}$ and $C_3 = \{3, 5, 6\}$. It can be shown that no variable can be fixed by applying Propositions 3 and 4 to constraints (17) and (18) taking p = 1. However, by applying claim (1) of Proposition 3 and claim (3) of Proposition 4 to constraints (17) and (18) taking $\lambda_1 = 2$ and $\lambda_2 = 1$ it follows that $x_1 = 1$ and $x_4 = x_5 = 0$ in every feasible solution to (P), since $z = -4x_1 - 2x_2 - x_3 + 2x_4 - x_5 - x_6$, $l_1 = -5$, $l_2 = -2$, $l_{z,C} = -7$ and $l_{1,1} = -1$. Therefore, fixing the variables x_1, x_4 and x_5, the constraint system (17)-(21) reduces to the constraint system (22)-(24).

$$2x_2 - 2x_3 + x_6 \leq 2 \quad (22)$$

$$-6x_2 + 3x_3 - 3x_6 \leq -4 \quad (23)$$

$$x_3 + x_6 \leq 1 \quad (24)$$

Let $C = \{C_4\}$, where $C_4 = \{3, 6\}$. By applying claim (1) of Proposition 3 and claim (5) of Proposition 4 to constraint (23) taking $\lambda_1 = 1$ it follows that $x_2 = 1$ and $x_3 = 0$ in every feasible solution to (P), since $z = -6x_2 + 3x_3 - 3x_6$, $l_4 = -3$ and $l_{z,C} = -9$. So, fixing the variables x_2 and x_3, the constraint system (22) - (24) reduces to the constraint system (25)-(26).

$$x_6 \leq 0 \quad (25)$$

$$-3x_6 \leq 2 \quad (26)$$

Let $C = \emptyset$. By applying claim (5) of Proposition 3 to constraint (25) it follows that $x_6 = 0$ in every feasible solution to (P).

6. CONCLUSIONS

In this paper we have presented a new procedure for obtaining lower bounds on linear functions that makes use of the information provided by certain families of packings, coverings and special ordered sets. It can determine better lower bounds than the traditional procedures, which do not consider either coverings or special ordered sets, and it can be particularly useful in problems without packings. We have also presented new methods for detecting infeasibility and fixing variables in 0-1 linear programming problems based on these lower bounds. They can detect some situations that the methods available in current literature cannot, since they allow consideration of several constraints jointly, whereas the existing methods consider only single constraints. Consequently, these new methods can improve the current preprocessing techniques.

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