STEADY STATE ANALYSIS OF AN M/D/1 QUEUE WITH TWO STAGES OF HETEROGENEOUS SERVER VACATIONS (M/D/G_1,G_2/1 QUEUE)

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ABSTRACT
A single server vacation queue with Poisson arrivals, deterministic service of constant duration b(> 0) and two stages of heterogeneous server vacations having different general (arbitrary) distributions is studied. This model is designated as (M/D/G_1,G_2/1). After completion of each service, the server may take a vacation with probability p or may continue working in the system with probability 1-p. Closed explicit forms for the steady state system size probability generation functions of various states of the server as well as the average number and the average waiting time in the system and the queue are obtained. Some new useful special cases including the known results of the M/D/1 queue are derived. Finally a numerical illustration is discussed.

Key words: Poisson arrivals, steady state, probability generating function, deterministic service, two-stage vacations, average system size, average waiting time.

MSC: 60 K 25

RESUMEN
Una cola con un servidor simple vacacional con arribos Poisson, con constante de duración de servicio determinística b(> 0) y servidor vacacional heterogéneo bietápico con distribuciones diferentes (arbitrarias) es estudiado. Este modelo es designado como M/D/G_1/G_2/1. Después de completar cada servicio el servidor puede tomar una vacación con probabilidad p o puede continuar trabajando en el sistema con probabilidad 1-p. Expresiones cerradas y explícitas para la función de probabilidad que genera el tamaño del sistema de estado estable para varios estados del servidor, así como el número promedio y el promedio del tiempo de espera en el sistema y la cola son obtenidos. Algunos casos nuevos especiales, incluyendo los resultados conocidos de colas M/D/1, son derivados. Finalmente una ilustración numérica es discutida.

Palabras clave: arribos de Poisson, estado firme, probabilidad de la función generadora, servicios determinísticos, vacaciones de dos-fase, tamaño del sistema de estado, promedio de tiempo de espera.

1. INTRODUCTION

Queueing systems such as M/D/1, D/M/1 and D/D/1 are widely found in queueing literature. (see Bunday [1986], Kashyap and Chaudhry [1988], Bhat [1972], etc). These systems assume a single server, Poisson/deterministic arrivals and exponential/deterministic service or deterministic arrivals and deterministic service. In all these models, the server is assumed to be always available in the system. However, in many real life situations the server may not always be available in the system. If the server is a human, he may leave the system from time to time and if the server is mechanical or electronic, it may breakdown from time to time. In the present paper, we have studied the steady state behaviour of a single server queue with deterministic service and two stages of heterogeneous server vacations having different general (arbitrary) distributions. For convenience we designate such a system as M/D/G_1,G_2/1.

Bernoulli schedule server vacations is used. This means that after each service the server may take a vacation with probability p or may continue staying in the system with probability 1-p. Such kind of policy including many other policies have been studied by many authors. To mention a few, we refer to Keilson and Servi [1986], Cramer [1989], Shanthikumar [1988] and Madan [1991, 1999]. For a complete overview of queues with vacation the reader is referred to Doshi [1980].

There are many situations in real life where the service times are constant. For example, a cycle of a washing machine takes a fixed length of time to complete one service and so does an air flight from a

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destination A to destination B. Another example is an automatic car-wash station where washing time for each car is constant. Such a system may be stopped (vacation) from time to time for its checking or overhauling etc after each service. Further, if the server is electronic or mechanical like the washing machine or the automatic car wash station, its first stage of vacation corresponds to waiting time till the expert repairmen arrive and the second stage may correspond to the actual repair time. Similarly if the server is a human, then his first stage vacation may be his actual vacation period and the second stage may be his travel time or the extra time he takes before he actually resumes work.

2. THE MATHEMATICAL MODEL

The mathematical model is described briefly by the following assumptions:

Poisson arrivals with mean arrival rate λ (>0) -

Deterministic server vacations with a constant service time b (>0) -

Bernoulli schedule server vacations which means that after each service the server may go on a vacation - with probability p and may stay on in the system with probability 1-p.

When the server takes a vacation, it consists of two stages of vacations with heterogeneous vacation - . Let $B_j(v)$ times, with the jth stage having probability density function $b_j(v)$ and the distribution function $\beta_j(x)dx$ be the conditional probability that the jth stage vacation will complete during the time interval (x,x+dx] given that the same was not complete till time x. Therefore,

$$\beta_j(x) = \frac{b_j(x)}{1 - B_j(x)}, \quad j = 1,2, \tag{1}$$

so that

$$b_j(v) = \beta_j(v)\exp\left[-\int_0^v \beta_j(x)dx\right], \quad j = 1,2, \tag{2}$$

DEFINITIONS AND NOTATIONS .3

We define

$H_n(t)$ = probability that at time t there are n (≥ 0) customers in the system and the server is providing service when n>0 and is idle but present in the system when n = 0.

$V_n^{(j)}(x,t)$ = probability that at time t there are n (≥ 0) customers in the system and the server is on j th stage of vacation (j = 1,2) with elapsed vacation time x. Correspondingly, $V_n^{(j)}(t) = \int_0^\infty V_n^{(j)}(x,t)dx$ is the probability that the server is on the jth stage of vacation (j = 1,2) without regard to the elapsed vacation time x.

Assuming that the steady state exists, we let $\lim_{t\to\infty} H_n(t) = H_n$, $\lim_{t\to\infty} V_n^{(j)}(x,t) = V_n^{(j)}(x)$, $\lim_{t\to\infty} V_n^{(j)}(t) = V_n^{(j)}(x) = \lim_{t\to\infty}\int_0^\infty V_n^{(j)}(x,t)dx = \int_0^\infty \lim_{t\to\infty} V_n^{(j)}(x,t)dx$ stand for the corresponding steady state probabilities.

Furthermore, define the following steady state probability generating functions:

(3a) $|z| < 1,$ $, \sum_{0}^{\infty} H_n z^n \; H(z) =$

(3b) $j = 1,2, \; |z| < 1,$ $\sum_{0}^{\infty} V_n^{(j)}(x) z^n, V^{(j)}(x,z) =$

(3c) $j = 1,2, \; |z| < 1,$ $\sum_{0}^{\infty} V_n^{(j)} z^n, V^{(j)}(z) =$

(3d) $|z| < 1.$ $, \sum_{0}^{\infty} k_n z^n = \sum_{0}^{\infty} e^{-\lambda b} \frac{(\lambda b)^n}{n!} z^n = e^{-\lambda b(1-z)} \; K(z) =$

STEADY STATE SYSTEM EQUATIONS .4

(4) $n \geq 0,$ $\sum_{i=2}^{n+1} H_i k_{n+1-i} + \int_{0}^{\infty} V_n^{(2)}(x)\beta_2(x)dx \; H_n = (1 - p)(H_0 + H_1)k_n + (1 - p)$

(5) $n \geq 1$ $\frac{\partial}{\partial x} V_n^{(1)}(x) + (\lambda + \beta_1(x))V_n^{(1)}(x) = \lambda V_{n-1}^{(1)}(x)$

(6) $, \frac{\partial}{\partial x} V_0^{(1)}(x) + (\lambda + \beta_1(x))V_0^{(1)}(x) = 0$

(7) $n \geq 1,$ $\frac{\partial}{\partial x} V_n^{(2)}(x) + (\lambda + \beta_2(x))V_n^{(2)}(x) = \lambda V_{n-1}^{(2)}(x)$

(8) $, \frac{\partial}{\partial x} V_0^{(2)}(x) + (\lambda + \beta_1(x))V_0^{(2)}(x) = 0$

Equations (4) to (8) are to be solved subject to the following boundary conditions:

(9) $n \geq 0,$ $V_n^{(1)}(0) = p(H_0 + H_1)k_n + p\sum_{i=2}^{n+1} H_i k_{n+1-i}$

(10) $n \geq 0.$ $V_n^{(2)}(0) = \int_{0}^{\infty} V_n^{(1)}(x)\beta_1(x)dx$

STEADY STATE PROBABILITY GENERATING FUNCTIONS .5

Multiplying both sides of equation (4) by z^{n+1}, sum over n from 0 to ∞ and use equation (3), we get

(11) $z\int_{0}^{\infty} V^{(2)}(x,z)\beta_2(x)dx. \; zH(z) = (1 - p)K(z)H(z) + (1 - p)(z - 1)K(z)H_0 +$

Replacing K(z) by $e^{-\lambda(1-z)}$ and simplify, then

(11a) $z\int_{0}^{\infty} V^{(2)}(x,z)\beta_2(x)dx. \; [z - (1 - p)e^{-\lambda b(1 - z)}]H(z) = (1 - p)(z - 1)e^{-\lambda b(1 - z)}H_0 +$

Next, by multiplying both sides of equation (5) by z^n, sum over n from 1 to ∞ and add the result to equation (6), use equation (3) and simplify, we get

(12) $$\frac{\partial}{\partial x}V^{(1)}(x,z)+(\lambda-\lambda z+\beta_1(x))V^{(1)}(x,z)=0.$$

A similar operation on equations (7) and (8) yields

(13) $$\frac{\partial}{\partial x}V^{(2)}(x,z)+(\lambda-\lambda z+\beta_2(x))V^{(2)}(x,z)=0.$$

Also, by multiplying both sides of equation (9) and (10) by z^{n+1} and z^n respectively and summing over n from 0 to ∞, using equation (3) and simplifying we obtain

(14) $$zV^{(1)}(0,z) = pH(z)K(z) + p(z - 1)K(z)H_0,$$

(14a) $$\int_0^\infty V^{(1)}(x,z)\beta_1(x)dx.\,V^{(2)}(0,z) =$$

Substituting for $K(z) = e^{-\lambda b(1 - z)}$ from (3d), into equation (14) then

(15) $$zV^{(1)}(0,z) = pH(z)e^{-\lambda b(1 - z)} + p(z - 1)e^{-\lambda b(1 - z)}H_0.$$

Integrating equations (12) and (13) between 0 and x then

(16) $$-\int_0^x \beta_1(t)dt,\,V^{(1)}(x,z) = V^{(1)}(0,z)\exp[-\lambda - \lambda z)x]$$

(17) $$-\int_0^x \beta_2(t)dt.\,V^{(2)}(x,z) = V^{(2)}(0,z)\exp[-(\lambda - \lambda z)x]$$

Again by integrating equations (16) and (17) by parts with respect to x we get

(18) $$\left[\frac{1-\bar{b}_1(\lambda-\lambda z)}{\lambda-\lambda z}\right],V^{(1)}(z) = V^{(1)}(0,z)$$

(19) $$\left[\frac{1-\bar{b}_2(\lambda-\lambda z)}{\lambda-\lambda z}\right],V^{(2)}(z) = V^{(2)}(0,z)$$

is the Laplace transform of $b_j(x)$, j = 1, 2. $\bar{b}_j(\lambda-\lambda z)=\int_0^\infty e^{-(\lambda-\lambda z)x}b_j(x)dx$ where

Using equation (15) in (18) and (14a) in (19) we obtain

(20) $$V^{(1)}(z)=\tfrac{1}{z}\left[pH(z)e^{-\lambda b(1-z)}+p(z-1)e^{-\lambda b(1-z)}H_0\right]\left[\frac{1-\bar{b}_1(\lambda-\lambda z)}{\lambda-\lambda z}\right],$$

(21) $$V^{(2)}(z)=\left[\int_0^\infty V^{(1)}(x,z)\beta_1(x)dx\right]\left[\frac{1-\bar{b}_2(\lambda-\lambda z)}{\lambda-\lambda z}\right].$$

appearing in equations (21) $\int_0^\infty V^{(2)}(x,z)\beta_{21}(x)dx$ and $\int_0^\infty V^{(1)}(x,z)\beta_1(x)dx$ In order to determine the integrals and (11a) respectively, multiply equation (16) by $\beta_1(x)$ and (17) by $\beta_2(x)$ and integrate both between 0 and x, thus

$$\int_0^\infty V^{(1)}(x,z)\beta_1(x)dx = V^{(1)}(0,z)\bar{b}_1(\lambda-\lambda z), \tag{23}$$

$$\int_0^\infty V^{(2)}(x,z)\beta_2(x)dx = V^{(2)}(0,z)\bar{b}_2(\lambda-\lambda z). \tag{24}$$

Using equation (24) in (11a) we obtain

$$\left[z-(1-p)e^{-\lambda b(1-z)}\right]H(z) = (1-p)(z-1)e^{-\lambda b(1-z)}W_0 + zV^{(2)}(0,z)\bar{b}_2(\lambda-\lambda_z), \tag{25}$$

and by using (23) in (21) we obtain

$$V^{(2)}(z) = \left[\frac{1-\bar{b}_2(\lambda-\lambda_z)}{\lambda-\lambda_z}\right]\bar{b}_1(\lambda-\lambda_z)V^{(1)}(0,z). \tag{26}$$

Next, substitute for $V^{(1)}(0,z)$ from equation (14) into equation (26) then

$$V^{(2)}(z) = \tfrac{1}{z}\left[pH(z)e^{-\lambda b(1-z)} + p(z-1)e^{-\lambda b(1-z)}H_0\right]\bar{b}_1(\lambda-\lambda_z)\left[\frac{1-\bar{b}_2(\lambda-\lambda_z)}{\lambda-\lambda_z}\right]. \tag{27}$$

Now, from (14a)

$$V^{(2)}(0,z) = \int_0^\infty V^{(1)}(x,z)\beta_1(x)dx = V^{(1)}(0,z)\bar{b}_1(\lambda-\lambda z).$$

which by using (14) it becomes

$$.\ V^{(2)}(0,z) = \tfrac{1}{z}\left[pe^{-\lambda b(1-z)} + p(z-1)e^{-\lambda b(1-z)}H_0\right]\bar{b}_1(\lambda-\lambda z) \tag{28}$$

Also, substituting for $V^{(2)}(0,z)$ from equation (28) into equation (25) then

$$\left[z-(1-p)e^{-\lambda b(1-z)}\right]H(z) = \left[pH(z)e^{-\lambda b(1-z)} + p(z-1)e^{-\lambda b(1-z)}H_0\right]\bar{b}_1(\lambda-\lambda z)\bar{b}_2(\lambda-\lambda z) + (1-p)(z-1)e^{-\lambda b(1-z)}H_0 \tag{29}$$

Furthermore, equation (29) can be re-written as

$$\left[p\bar{b}_1(\lambda-\lambda z)\bar{b}_2(\lambda-\lambda z)+(1-p)\right](z-1)e^{-\lambda b(1-z)}H_0 = \left[z-(1-p)e^{-\lambda b(1-z)} - \bar{b}_1(\lambda-\lambda z)\bar{b}_2(\lambda-\lambda z)pe^{-\lambda b(1-z)}\right]H(z) \tag{30}$$

which yields

$$\frac{\left[p\bar{b}_1(\lambda-\lambda z)\bar{b}_2(\lambda-\lambda z)+(1-p)\right](z-1)e^{-\lambda b(1-z)}H_0}{z-(1-p)e^{-\lambda b(1-z)} - p\bar{b}_1(\lambda-\lambda z)\bar{b}_2(\lambda-\lambda z)e^{-\lambda b(1-z)}}.\ H(z) = \tag{31}$$

Substituting for H(z) from equation (31) into equation (20) and (27) and simplifying we obtain

$$(32)\qquad \left[\frac{1-\bar{b}_1(\lambda-\lambda z)}{\lambda-\lambda z}\right]\left[\frac{p(z-1)e^{-\lambda b(1-z)}H_0}{z-(1-p)e^{-\lambda b(1-z)}-p\bar{b}_1(\lambda-\lambda z)\bar{b}_2(\lambda-\lambda z)e^{-\lambda b(1-z)}}\right], V^{(1)}(z) =$$

$$(33)\qquad \left[\frac{1-\bar{b}_2(\lambda-\lambda z)}{\lambda-\lambda z}\right]\left[\frac{p(z-1)e^{-\lambda b(1-z)}\bar{b}_1(\lambda-\lambda z)H_0}{z-(1-p)e^{-\lambda b(1-z)}-p\bar{b}_1(\lambda-\lambda z)\bar{b}_2(\lambda-\lambda z)e^{-\lambda b(1-z)}}\right]. V^{(2)}(z) =$$

It remains to determine the only unknown constant H_0 which appears in the numerators of the right hand sides of equations (31), (32) and (33). In order to achieve this objective, use the normalizing condition $H(1) + V^{(1)}(1) + V^{(2)}(1) = 1$. However, since each of the equations (31), (32) and (33) are indeterminate of the zero/zero form, we employ L'Hopital's rule and simplify to obtain

$$(34)\qquad \lim_{z\to 1} H(z) = \frac{H_0}{1-\lambda(b+pE(V_1)+pE(V_2))}, H(1) =$$

$$(35)\qquad \lim_{z\to 1} V^{(1)}(z) = \frac{pE(V_1)H_0}{1-\lambda(b+pE(V_1)+pE(V_2))}, V^{(1)}(1) =$$

$$(36)\qquad \lim_{z\to 1} V^{(2)}(z) = \frac{pE(V_2)H_0}{1-\lambda(b+pE(V_1)+pE(V_2))}. V^{(2)}(1) =$$

Using equations (34), (35) and (36) and simplifying, the normalizing condition yields

$$(37)\qquad H_0 = \frac{1-\lambda(b+pE(V_1)+pE(V_2))}{1+pE(V_1)+pE(V_2)}.$$

Equation (37) yields the condition

$$(38)\qquad \lambda(b + pE(V_1) + pE(V_2)) < 1$$

under which the steady state shall exist.

Note that if there are no server vacations then letting p = 0, equations (37) and (38) yield $H_0 = 1 - \lambda b$, $\lambda b < 1$ which are known results of the M/D/1 queue.

Now let H, $V^{(1)}$ and $V^{(2)}$ denote the steady state probabilities that the server is present in the system, the server is on first stage of vacation and the server is on second stage of vacation respectively. Then using (37) in equations(34), (35) and (36), we obtain

$$(39)\qquad H = H(1) = \frac{1}{1+pE(V_1)+pE(V_2)},$$

$$(40)\qquad \frac{pE(V_2)}{1+pE(V_1)+pE(V_2)}, V^{(1)} = V^{(1)}(1) =$$

$$(41)\qquad \frac{pE(V_1)}{1+pE(V_1)+pE(V_2)}. V^{(2)} = V^{(2)}(1) =$$

Next, to find the utilization factor ρ, note that since H, the probability that the server is present in the system also includes H_0, the probability that the server is idle, then by using (37) and (39), the system's utilization factor ρ is given by

(42) $$\frac{\lambda(b + pE(V_1) + pE(V_2))}{1 + pE(V_1) + pE(V_2)} . \rho = H - H_0 =$$

Substituting the value of H_0 from (37) into (31), (32) and (33) we have now completely determined all the probability generating functions.

THE AVERAGE SYSTEM SIZE AND THE AVERAGE WAITING TIME IN THE SYSTEM AND THE QUEUE .6

We define $P(z) = H(z)+V^{(1)}(z) + V^{(2)}(z)$ to be the probability generating function of the system size regardless of the state of the server. Then on adding (31), (32), (33) we have

$$\frac{N(z)}{D(z)} P(z) =$$

where

$$\left\{ \left[p\bar{b}_1(\lambda - \lambda z)\bar{b}_2(\lambda - \lambda z) + (1 - p)\right] (z - 1)e^{-\lambda b(1-z)} \right. N(z) =$$

(43) $$\left. \frac{p}{\lambda}\left[\bar{b}_1(\lambda - \lambda z) - 1\right]e^{-\lambda(z-1)} + \frac{p}{\lambda}\bar{b}_1(\lambda - \lambda z)\left[\bar{b}_2(\lambda - \lambda z) - 1\right]e^{-\lambda(1-z)} \right\} H_0 \quad +$$

(44) $$e^{-\lambda b(1-z)} - p\bar{b}_1(\lambda - \lambda z)\bar{b}_2(\lambda - \lambda z)e^{-\lambda b/1-z)} D(z) = z - (1 - p)$$

where H_0 has been found in equation (37).

at z = 1 . However, since P(z) is indeterminate of the $\frac{d}{dz}P(z)$ Then the average system size is given by L = zero/zero form at z = 1, we employ L'Hopital's rule twice and obtain

(45) $$\frac{D'(1)N''(1) - N'(1)D''(1)}{2[D + (1)]^2} L =$$

where dashes denote derivatives with respect to z at z = 1.

Carrying out the desired derivatives and after a lot of algebra and simplification we obtain

(46) $$N'(1) = [1 + E(V_1) + pE(V_2)]H_0$$

(47) $$E(V_1^2) + E(V_2^2))]H_0, N''(1) = [2\lambda b(1 + pE(V_1) + pE(V_2)) + \lambda b(2E(V_1) + 2E(V_2) + 2E(V_1)E(V_2) +$$

(48) $$D'(1) = 1 - \lambda b - p\lambda E(V_1) - p\lambda E(V_2)$$

(49) $$\left\{\lambda^2 b^2 + 2p\lambda^2 b(E(V_1) + E(V_2)) + 2p\lambda^2 E(V_1)E(V_2) + p\lambda^2(E(V_1^2) + E(V_2^2))\right\} D''(1) = -$$

are respectively the mean vacation times and the second moments of $E(V_1^2), E(V_2^2)$ where $E(V_1)$,$E(V_2)$ and the stage 1 and stage 2 vacation times. We further note that in carrying out the above derivatives, we have j = 1,2. $\bar{b}_j''(\lambda - \lambda z) = \lambda^2 E(V_j^2)$, and $\bar{b}_1(\lambda - \lambda z) = 1$, $\bar{b}_j'(\lambda - \lambda z) = \lambda E(V_j)$ used the facts that at z = 1,

Using (46), (47), (48) (49) into equation (45) the average system size L is explicitly determined.

Furthermore, by using Little's formulas, we can obtain the average waiting time in the system as

(50) $$\frac{L}{\lambda} W =$$

where L has been found in (45).

In addition, we obtain the average queue size L_q and the average waiting time in the queue W_q as L_q= L - ρ,

(51) $$\frac{Lq}{\lambda}\ W_q\ =$$

where L and ρ have been found in equations (45) and (42) respectively.

SPECIAL CASES .7

No Server Vacations Case 1:

In this case, we let p = 0 in the main results obtained in equations (31), (32), (33) and equations (45) to (51). We thus obtain

$$V^{(1)}(z) = 0 = V^{(2)}(z)$$

(52) $$\frac{(z-1)e^{-\lambda b(1-z)}(1-\lambda b)}{z-e^{-\lambda b(1-z)}} H(z) =$$

(53) $$\frac{2b-\lambda b^2}{2(1-\lambda b)} W = \qquad \frac{2\lambda b-\lambda^2 b^2}{2(1-\lambda b)}, L =$$

(54) $$\frac{\lambda b^2}{2(1-\lambda b)}\ W_q = \qquad \frac{\lambda^2 b^2}{2(1-\lambda b)}, L_q =$$

We note that results in (52), (53), (54) agree with known results of the M/D/1 queue. (see Kashyap and Chaudhry [5], page 60)

No Second Stage of Vacation Case 2:

in the main results and obtain $\frac{1}{\beta_2} \to 0$ take limit as $\bar{b}_2(\lambda-\lambda z)=0, E(V_2)=0, E(V_2^2)=0,$ In this case we let =

$V^{(2)}(z)$ = 0 and

(55) $$\frac{[p\bar{b}_1(\lambda-\lambda z)+(1-p)](z-1)e^{-\lambda b(1-z)}H_0}{z-(1-p)e^{-\lambda b(1-z)}-p\bar{b}_1(\lambda-\lambda z)e^{-\lambda b(1-z)}}, H(z) =$$

(56) $$\left[\frac{1-\bar{b}_1(\lambda-\lambda z)}{\lambda-\lambda z}\right]\left[\frac{p(z-1e^{-\lambda b(1-z)}H_0}{z-(1-p)e^{-\lambda b(1-z)}-p\bar{b}_1(\lambda-\lambda z)e^{-\lambda b(1-z)}}\right] V^{(1)}(z) =$$

where now we have from the main results

(57) $$H_0 = \frac{1-\lambda(b+pE(V_1))}{1+pE(V_1)}$$

(58) $$N'(1) = 1 - \lambda(b + pE(V_1)),$$

(59) $\quad , \quad E(V_1^2))\, N''(1) = 2\lambda b(1 + pE(V_1)) + \lambda p(2E(V_1) +$

(60) $\quad D'(1) = 1 - \lambda b - p\lambda E(V_1)$

(61) $\quad \left\{\lambda^2 b^2 + 2p\lambda^2 bE(V_1) + p\lambda^2 E(V_1^2))\right\} \qquad D''(1) = -$

so that in this case we have

$$\frac{[1-\lambda b - p\lambda E(V_1)[2\lambda b(1+pE(V_1)) + \lambda p(2E(V_1)+E(V_1^2)]}{2[1-\lambda b - p\lambda E(V_1)]^2}\ L =$$

(62)
$$+\frac{[1+pE(V_1)]H_0[\lambda^2 b^2 + 2p\lambda^2 bE(V_1) + p\lambda^2 E(V_1^2)]}{2[1-\lambda b - p\lambda E(V_1)]^2}$$

Further we can also find W, L_q and W_q as before.

First Stage Vacation is Exponential and No Second Stage Vacation Case 3:

, in the results of case 2 and obtain $E(V_1) = \frac{1}{\beta_1}, E(V_1^2) = \frac{2}{\beta_1^2}$ and $\bar{b}_1(\lambda - \lambda z) = \frac{\beta_1}{\lambda - \lambda z + \beta_1}$ In this case we let

(63)
$$\frac{p\left(\frac{\beta_1}{\lambda - \lambda z + \beta_1}\right) + (1-p)(z-1)e^{-\lambda b(1-z)}\left\{\frac{[1-\lambda(b+\frac{p}{\beta_1})]}{1+\frac{p}{\beta_1}}\right\}}{z-(1-p)e^{-\lambda b(1-z)} - p\,(\frac{\beta_1}{\lambda-\lambda z+\beta_1})\,e^{-\lambda b(1-z)}},\ H(z) =$$

(64)
$$\left[\frac{1}{\lambda - \lambda z + \beta_1}\right]\left[\frac{p(z-1)e^{-\lambda b(1-z)}\left\{\frac{[1-\lambda(b+\frac{p}{\beta_1})]}{1+\frac{p}{\beta_1}}\right\}}{z-(1-p)e^{-\lambda b(1-z)} - p\,(\frac{\beta_1}{\lambda-\lambda z+\beta_1})\,e^{-\lambda b(1-z)}}\right],\ V^{(1)}(z) =$$

(65)
$$\frac{[1+\frac{p}{\beta_1}]H_0\left[\lambda^2 b^2 + \frac{2p\lambda^2 b}{\beta_1} + \frac{2\lambda^2}{\beta_1^2}\right]}{2[1-\lambda b - \frac{p\lambda}{\beta_1}]^2}.\ + \frac{[1-\lambda b - \frac{p\lambda}{\beta_1}][2\lambda b(1+\frac{p}{\beta_1}) + 2\lambda p(\frac{1}{\beta_1}+\frac{1}{\beta_1^2})]H_0}{2[1-\lambda b - \frac{p\lambda}{\beta_1}]^2}\ L =$$

Both Vacation Stages are Heterogeneous Exponential Case 4:

$E(V_1) = \frac{1}{\beta_1}, E(V_1^2) = \frac{2}{\beta_1^2}, E(V_2) = \frac{1}{\beta_2}, \quad \bar{b}_1(\lambda - \lambda z) = \frac{\beta_1}{\lambda - \lambda z + \beta_1}, \bar{b}_2(\lambda - \lambda z) = \frac{\beta_2}{\lambda - \lambda z + \beta_2}$, In this case we let

in the main results and obtain $E(V_2^2) = \frac{2}{\beta_2^2}$ and

(66)
$$,\ \frac{1-\lambda\left(b+\frac{p}{\beta_1}+\frac{p}{\beta_2}\right)}{1+\frac{p}{\beta_1}+\frac{p}{\beta_2}},\ \lambda\left(b+\frac{p}{\beta_1}+\frac{p}{\beta_2}\right) < 1\, H_0 =$$

(67)
$$\rho = \frac{\lambda\left(b+\frac{p}{\beta_1}+\frac{p}{\beta_2}\right)}{1+\frac{p}{\beta_1}+\frac{p}{\beta_2}},$$

(68) $$\frac{1}{1+\frac{p}{\beta_1}+\frac{p}{\beta_2}},\quad V^{(1)}=\frac{\frac{p}{\beta_2}}{1+\frac{p}{\beta_1}+\frac{p}{\beta_2}},\ V^{(2)}=\frac{\frac{p}{\beta_1}}{1+\frac{p}{\beta_1}+\frac{p}{\beta_2}}.\ H=$$

(69) $$\frac{\left[\frac{p\beta_1\beta_2}{(\lambda-\lambda z+\beta_1)(\lambda-\lambda z+\beta_2)}+(1-p)\right](z-1)e^{-\lambda b(1-z)}\left\{\frac{[\beta_1\beta_2-\lambda(b\beta_1\beta_2+p\beta_1+p\beta_2)]}{\beta_1\beta_2+p\beta_1+p\beta_2}\right\}}{z-(1-p)e^{-\lambda b(1-z)}-\frac{p\beta_1\beta_2}{(\lambda-\lambda z+\beta_1)(\lambda-\lambda z+\beta_2)}e^{-\lambda b(1-z)}},\ H(z)=$$

(70) $$\left[\frac{1}{\lambda-\lambda z+\beta_1}\right]\left[\frac{p(z-1)e^{-\lambda p(1-z)}\left\{\frac{[\beta_1\beta_2-\lambda(b\beta_1\beta_2+p\beta_1+p\beta_2)]}{\beta_1\beta_2+p\beta_1+p\beta_2}\right\}}{z-(1-p)e^{-\lambda b(1-z)}-\frac{p\beta_1\beta_2}{(\lambda-\lambda z+\beta_1)(\lambda-\lambda z+\beta_2)}e^{-\lambda b(1-z)}}\right],\ V^{(1)}(z)=$$

(71) $$\left[\frac{1}{\lambda-\lambda z+\beta_2}\right]\left[\frac{p(z-1)e^{-\lambda p(1-z)}\left(\frac{\beta_1}{\lambda-\lambda z+\beta_1}\right)\left\{\frac{[\beta_1\beta_2-\lambda(b\beta_1\beta_2+p\beta_1+p\beta_2)]}{\beta_1\beta_2+p\beta_1+p\beta_2}\right\}}{z-(1-p)e^{-\lambda b(1-z)}-\frac{p\beta_1\beta_2}{(\lambda-\lambda z+\beta_1)(\lambda-\lambda z+\beta_2)}e^{-\lambda b(1-z)}}\right],\qquad V^{(2)}(z)=$$

$$\frac{\left[1-\lambda b-\frac{p\lambda}{\beta_1}-\frac{p\lambda}{\beta_2}\right]\left[2\lambda b\left(1+\frac{p}{\beta_1}+\frac{p}{\beta_2}\right)+2\lambda b\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}+\frac{1}{\beta_1\beta_2}+\frac{1}{\beta_1^2}+\frac{1}{\beta_2^2}\right)\right]H_0}{2\left[1-\lambda b-\frac{p\lambda}{\beta_1}-\frac{p\lambda}{\beta_2}\right]^2}\ L=$$

(72) $$\frac{\left[1-\frac{p}{\beta_1}-\frac{p}{\beta_2}\right]H_0\left[\lambda^2b^2+2p\lambda^2b\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)+2p\lambda^2\left(\frac{1}{\beta_1\beta_2}+\frac{1}{\beta_1^2}+\frac{1}{\beta_2^2}\right)\right]}{2\left[1-\lambda b-\frac{p\lambda}{\beta_1}-\frac{p\lambda}{\beta_2}\right]^2}\qquad +$$

Using ρ found in equation (67) and L in equation (76) we can further find

(73) $$.\ W_q=\frac{L_q}{\lambda}\qquad L_q=L-\rho,\qquad W=\frac{L}{\lambda},$$

2-Erlangian Vacations Case 5:
(Both Vacation Stages are Identically Exponential)

In this case, we let $\beta_1=\beta_2=\beta$ in the results of case 4 and obtain

(74) $$,\quad H_0=\frac{1-\lambda\left(b+\frac{2p}{\beta}\right)}{1+\frac{2p}{\beta}},\quad \lambda\left(b+\frac{2p}{\beta}\right)<1$$

(75) $$\rho=\frac{\lambda\left(b+\frac{2p}{\beta}\right)}{1+\frac{2p}{\beta}},$$

(76) $$H = \frac{1}{1+\frac{2p}{\beta}}, \quad V^{(1)} = \frac{\frac{p}{\beta}}{1+\frac{2p}{\beta}}, \; V^{(2)} = \frac{\frac{p}{\beta}}{1+\frac{2p}{\beta}},$$

(77) $$H(z) = \frac{\left\{p\left(\frac{\beta}{\lambda-\lambda z+\beta}\right)^2 + (1-p)\right\}(z-1)e^{-\lambda b(1-z)}\left\{\frac{\beta-\lambda(b\beta+p)}{\beta+2p}\right\}}{z-(1-p)e^{-\lambda b(1-z)} - p\left(\frac{\beta}{\lambda-\lambda z+\beta}\right)^2 e^{-\lambda b(1-z)}},$$

(78) $$V^{(1)}(z) = \left[\frac{1}{\lambda-\lambda z+\beta}\right]\left[\frac{p(z-1)e^{-\lambda b(1-z)}\left\{\frac{[\beta-\lambda(b\beta+p)]}{\beta+2p}\right\}}{z-(1-p)e^{-\lambda b(1-z)} - p\left(\frac{\beta}{\lambda-\lambda z+\beta}\right)^2 e^{-\lambda b(1-z)}}\right],$$

(79) $$V^{(2)}(z) = \left[\frac{1}{\lambda-\lambda z+\beta}\right]\left[\frac{p(z-1)e^{-\lambda b(1-z)}\left(\frac{\beta}{\lambda-\lambda z+\beta}\right)\left\{\frac{[\beta-\lambda(b\beta+p)]}{\beta+2p}\right\}}{z-(1-p)e^{-\lambda b(1-z)} - p\left(\frac{\beta}{\lambda-\lambda z+\beta}\right)^2 e^{-\lambda b(1-z)}}\right],$$

(80) $$\frac{\left[1+\frac{2p}{\beta}\right]H_0\left[\lambda^2 b^2 + \frac{2p\lambda^2 b}{\beta} + \frac{6p\lambda^2}{\beta^2}\right]}{2\left[1-\lambda b-\frac{2p\lambda}{\beta}\right]^2} + L = \frac{\left[1-\lambda b-\frac{2p}{\beta}\right]\left[2\lambda b\left(1+\frac{2p}{\beta}\right)+2\lambda p\left(\frac{2}{\beta}+\frac{3}{\beta^2}\right)\right]H_0}{2\left[1-\lambda b-\frac{2p\lambda}{\beta}\right]^2}$$

A NUMERICAL ILLUSTRATION .8

In order to see the effect of various parameters on server's idle time H_0, system's utilization factor ρ, the proportion of time the server is present in the system, the proportion of server's vacation time in stage 1 and stage 2 and various other queue characteristics such as L, W, L_q and W_q, we base our numerical example on the results given in case 4. For this purpose, arbitrary values of λ, p, b, β_1 and β_2 are chosen, as indicated in the following tables, such that the steady state condition

$$< 1\,\lambda\left(b+\frac{p}{\beta_1}+\frac{p}{\beta_2}\right)$$

given in (66) is always satisfied.

Table 1. Computed values of the probabilities of various states $\beta_1 = 20$, $\beta_2 = 30$. $\frac{1}{20}$, of the server for fixed b =

Probability (p)	H	$V^{(1)}$	$V^{(2)}$
0.00	1.000	0.00000	0.00000
0.10	0.992	0.00331	0.00496
0.20	0.984	0.00656	0.00984
0.30	0.976	0.00976	0.01463
0.40	0.968	0.01290	0.01936
0.50	0.960	0.01600	0.02400
0.60	0.952	0.01905	0.02857
0.70	0.945	0.02205	0.03307
0.80	0.938	0.02500	0.03750
0.90	0.930	0.02791	0.04186
1.00	0.923	0.03080	0.04615

$\beta_1 = 20$, $\beta_2 = 30.\frac{1}{20}$, **Table 2**. Computed values of the various queue characteristics for fixed, b =

	p	Ho	ρ	L	W	Lq	Wq
λ = 2	0.00	0.900	0.100	0.10556	0.05280	0.00556	0.00278
	0.10	0.876	0.116	0.12752	0.06375	0.01182	0.00591
	0.20	0.853	0.131	0.14935	0.07468	0.01815	0.00908
	0.30	0.829	0.146	0.17108	0.08554	0.02478	0.01239
	0.40	0.807	0.161	0.19274	0.09637	0.03144	0.01572
	0.50	0.784	0.176	0.21430	0.10716	0.03830	0.01915
	0.60	0.762	0.191	0.23585	0.11793	0.04537	0.02268
	0.70	0.740	0.205	0.25740	0.12868	0.05270	0.02635
	0.80	0.719	0.219	0.27890	0.13943	0.06015	0.03007
	0.90	0.698	0.233	0.30037	0.15018	0.06781	0.03391
	1.00	0.677	0.246	0.32192	0.16096	0.07572	0.03786
λ = 4	0.00	0.800	0.200	0.22500	0.05600	0.02500	0.00600
	0.10	0.760	0.231	0.28095	0.07000	0.05000	0.01200
	0.20	0.721	0.262	0.33821	0.08500	0.07600	0.01900
	0.30	0.683	0.293	0.39707	0.09900	0.10400	0.02600
	0.40	0.645	0.323	0.45790	0.11400	0.13500	0.03400
	0.50	0.608	0.352	0.52101	0.13000	0.16900	0.04200
	0.60	0.571	0.381	0.58700	0.14700	0.20600	0.05100
	0.70	0.535	0.410	0.65640	0.16400	0.24700	0.06200
	0.80	0.500	0.438	0.73000	0.18300	0.29300	0.07300
	0.90	0.465	0.465	0.80870	0.20200	0.34400	0.08600
	1.00	0.431	0.492	0.89385	0.22300	0.40200	0.10000
λ = 6	0.00	0.700	0.300	0.36400	0.06100	0.06400	0.01100
	0.10	0.645	0.347	0.47400	0.07900	0.12700	0.02100
	0.20	0.590	0.393	0.59300	0.09900	0.20000	0.03300
	0.30	0.537	0.439	0.72300	0.12000	0.28400	0.04700
	0.40	0.484	0.484	0.86800	0.14500	0.38400	0.06400
	0.50	0.432	0.528	1.03300	0.17200	0.50500	0.08400
	0.60	0.381	0.571	1.22600	0.20400	0.65500	0.10900
	0.70	0.331	0.614	1.46000	0.24300	0.84600	0.14100
	0.80	0.281	0.656	1.75500	0.29300	1.09900	0.18300
	0.90	0.233	.698	2.14900	0.35800	1.45100	0.24200
	1.00	0.185	0.739	2.71600	0.45300	1.97800	0.33000

It is clear from Table 1 that, as expected, when p increases the proportion of time that the server is present in the system (H) decreases but both $V^{(1)}$ and $V^{(2)}$ increase. Also, from Table 2 all given system characteristics are varying with p and λ. In particular for fixed p when λ increases the server's idle time (Ho) decreases but all other quantities ρ, L, Lq, W and Wq increase as it should be. Similar conclusion can be drawn when λ is held fixed and p increases.

Table 3(a). Computed values of various states of the server for fixed $b = \frac{1}{20}$, p = 0.5 and λ = 6.

β_1	β_2	H	$V^{(1)}$	$V^{(2)}$
22	32	0.963	0.01505	0.02189
	36	0.965	0.01340	0.02193
	40	0.966	0.01208	0.02195
26	32	0.966	0.01510	0.01858
	36	0.968	0.01344	0.01861
	40	0.969	0.01212	0.01864
30	32	0.969	0.01514	0.01615
	36	0.970	0.01348	0.01617
	40	0.972	0.01215	0.01619

Table 3(b). Computed values of various queue characteristics for fixed $b = \frac{1}{20}$, p = 0.5 and λ = 6.

β_1	β_2	Ho	ρ	L	W	Lq	Wq
22	32	0.453	0.511	0.94816	0.15803	0.43763	0.07294
	36	0.463	0.501	0.90838	0.15139	0.40704	0.06784
	40	0.472	0.494	0.87812	0.14635	0.38415	0.06403
26	32	0.474	0.492	0.86632	0.14439	0.37433	0.06239
	36	0.485	0.483	0.82987	0.13831	0.34714	0.05786
	40	0.494	0.475	0.80211	0.13369	0.32681	0.05447
30	32	0.490	0.478	0.81175	0.13529	0.33345	0.05575
	36	0.501	0.469	0.77747	0.12958	0.30847	0.05141
	40	0.510	0.462	0.75134	0.12522	0.28980	0.04830

Table 3(a) and 3(b) show that for fixed**,** when β_1, when β_2 increases (which means that $1/\beta_2$, the mean vacation time in stage 2 decreases), H increases, $V^{(1)}$, $V^{(2)}$, L, W, Lq and Wq all decrease as it should be. A similar conclusion can be drawn when β_1 varies and β_2 is held fixed.

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