A NOTE ON THE PREDICTION AND TESTING OF SYSTEM RELIABILITY UNDER SHOCK MODELS

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ABSTRACT
The realibility of finite systems is studied. The functioning of a system is characterized by a vector of binary variables. The a priori distribution of the random vector permits to establish a linear probability model. It is used for deriving a predictor of the realibility and its error. Under a set of mild conditions T-Student based inferences can be made when a sufficiently large sample size is uded. The independent case and a shock dependent model are studied.

Key words and phrases: realibility, Bayesian procedures, prediction.

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RESUMEN
La fiabilidad de sistemas finitos es estudiada. Su funcionamiento es caracterizado por vectores de variables binarias. La distribución a priori de los vectores aleatorios permite establecer un modelo probabilístico lineal. Este es utilizado para derivar un predictor de la fiabilidad y de su error. Bajo un conjunto de condiciones suaves se pueden realizar inferencias basadas en la T-Student cuando el tamaño de la muestra es suficientemente grande.

Palabras clave y frases: fiabilidad, procedimiento Bayesiano, predicción.

1. INTRODUCTION

A systemic analysis, of different problems, provides a theoretic frame that allows to use common techniques for predicting the probability that it works. We will assume that we cope with a finite set of entities which are clustered in components. A certain function, evaluated in the observed working condition of the entities, evaluated the functioning of the system. For each evaluation of the function a set of weights can be computed. The realibility can be evaluated when the involved probabilities are known, which is the common case. The a priori information permits to use Bayes Theorem and some predictors are proposed.

Section 2 is devoted to the description of the system and of the realibility of its functioning. The independence of the components is assumed. Under a linear probability model a predictor is proposed. It is model umbiased. If a sufficiently large number or realizations of the system are observed the related test statistic is a T-test.

Section 3 introduces the notion of shock. The independence is not longer valid. The realibility can be predicted when an adequate random experiment is performed. Again T-Student based inferences can be implemented.

Section 4 discusses some examples, where this approach can be used. They model environmental, economic and technical problems.

2. MODELING THE FUNCTIONING

Take a finite set $\zeta = \{1,...,n\}$ and a partition $\aleph = \{A\}$ defined on it, $[\cup_{A\in\aleph}A = \zeta$ $A \cap A' = \Phi$ if $A, A' \in \aleph]$,. Each $i \in A$ functions correctly (FC) with a probability p_i, $P = \{p_i: i \in A\}$. A set $A \in \aleph$ functions correctly (A - FC) if every $i \in A$ FC. Hence $\pi_A = \Pi_{i\in A}p_i$ is the probability that A - FC. The performance of i is characterized by a binary random variable

$$X_i = \begin{Bmatrix} 1 & \text{if} & i-FC \\ 0 & & \text{otherwise} \end{Bmatrix}$$

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It is a Bernoulli random variable.Then A - FC whenever

$X_A = \Pi_{i \in A} x_i = 1$

To analyze the functioning of a system $\Psi = \{\zeta, \phi\}$ is the objective of this paper.

The function

$$\phi(X) = \begin{Bmatrix} 1 & \text{if} & \zeta - FC \\ 0 & & \text{otherwise} \end{Bmatrix}$$

evaluates the status of the system where

$X = \{X_A: X_A = (X_1,...,X_{n_A})^T \& A \in \aleph\}$

n_A = number of components of A.

An adequate set of weights $W = \{\delta(A): A \in \aleph\}$ can be determined by solving the equation

$\phi(X) = \Sigma_{A \in \aleph} \delta(A) \Pi_{i \in A} X_i$

The function

$h(P) = \Sigma_{A \in \aleph} \delta(A) \Pi_{i \in A} p_i$

measures the realibility of the system if the components of A are independent. Note that this models, in technical problems, the fact that ζ represents a device with equivalent parallel components. Then it functions if at least one A works. Hence

$\delta(X_A) = 1$ for at least one $A \in \aleph \Rightarrow \zeta$ - FC

and the probability that it functions is given by $E[\phi(X)] = h(p)$.

Different economical, technical and social systems use weights, for their components, that are not necessarily equal to one. Engeland-Huseby (1991) denoted by h(P) the realibility of a network system. P is named 'realibility vector'. It may be unknown.

Generally the system is constructed but its performance must be studied. Hence, the decision maker (DM) observes a random X and evaluates $\phi(X)$, which is an unbiased estimate of h(P). Then its error is the variance

$V[\phi(X)] = \Sigma_{A \in \aleph} \delta^2(A) \Pi_{i \in A} p_i(1 - p_i)$

if the random variables X_i are mutually independent.

For obtaining the sample the Dm designs an experiment and $w_1,...,w_m$ random and independent events, from the corresponding probability space (Ω, σ, μ), are observed. We compute the sample estimates $\phi[X(w_1)],...,\phi[X(w_m)]$.

An estimator of the multilinear P-function h(P) is

$$1.\ \hat{h}(p) = \frac{\sum_{t=1}^{m} \phi[X(w_t)]}{m}$$

with error

1. $V\left[\hat{h}(P)\right] = \frac{1}{m}\sum_{A\in\aleph}\delta^2(A)\Pi_{i\in A}p_i(1-p_i)$.

Note that this estimate is more accurate than the previous one.

Experts can combine their opinions about the bahavior of the components of ζ. Gasimyr-Natuig (1996) analyzed this problem by using a Standard Bayes Theorem approach. Scott (1977) proposed a Bayesian method for finite population inferences. We will use both ideas for modeling a Bayesian approach to this problem.

Suppose that a vector of real numbers Z = $(Z_1,...,Z_N)$, the data (D), wich is related with X is completely or partially known. Using a sampling design d(o: Z), which depends on the Z - vector, we observe, the sample s. Gasymir-Natuig (1995) assumed that P is unknown but that the DM is able to model the uncertainty in terms of a prior distribution Q. The use of this prior yields

$$Q_D(Z{:}X(w)) = \begin{cases} \dfrac{d(s:Z)Q(X(w):Z)}{\int_\Omega d(s:Z)Q(X(w):Z)\partial\mu(X(w):Z)} & \text{if} \quad X(w)\in\Omega \\ 0 & \text{otherwise} \end{cases}$$

permits to use Bayes Theorem. w denotes a random event that identifies the actual population, μ is the Lebesgue measure and D denote the data set. If Z is available at the data analysis stage $Q_D(X(w){:}\ Z)$ is proportional to Q(X(w): Z). Note that

$$E[h(p)] = \int P[\Phi(X) = 1 : Z]Q_D(X(w : Z))$$

The sampling design is unimportant for the inferences. Hence the DM should only observe experimental points generated by some devise and to use the information provided by the data through the guessed prior.

Another method is to use the functional relation between X(w) and Z. The DM may postulate that for the prior Q[X(w)] the observations are iid random vectors distributed with

$$E(X(w) : Z) = \beta Z_t + \varepsilon$$

and

$$\mathrm{Cov}\left[X_t(w), X_{t'}(w){:}Z\right] = \begin{cases} \sigma_t^2 & \text{if} \quad t = t' \\ 0 & \text{otherwise} \end{cases}$$

In terms of our problem we can write the linear probability model by fixing A as an index:

(1) $X_A = \beta Z_A + \varepsilon_A$

where Z is a known variable, or at least it can be measured, β is an unknown parameter and ε is an unobservable random variable with model expectation

$$E_\mu(\varepsilon_A) = 0$$

and covariance

$$\mathrm{Cov}_\mu\left[\varepsilon_A, \varepsilon_{A'}\right] = \begin{cases} \sigma_A^2 = \Pi_{i\in A}p_i(1-p_i) = \Pi_{i\in A}\sigma_i^2 & \text{if} \quad A = A' \\ 0 & \text{otherwise} \end{cases}$$

under the hypothesis of independence of the model residuals.

The measurement of the Z_A's permits to predict

$P(X_A = 1) = \beta Z_A$

Note that we are dealing with is a Generalized Linear Model with binary random component and identity link function, see Agresti (1996).

The use of $\phi(X)$ as a predictor of h(p) was suggested previously. As the sampling design is non informative we can rely on the results of a reasonable experimental design for obtaining the required information.

Proposition 1. Suppose that a certain experimental design generates m random results of a system ζ with a model structure $\zeta = \Sigma_{A\in\aleph}$ A. Take the function $\phi_0[X(w_t)]$ and the model given by (1). The hypothesis that ζ is reliable $[h(P) \geq \gamma]$ can be tested by using

$$\frac{\hat{h}(P)-\gamma}{\sqrt{\frac{1}{m}\sum_{a\in\aleph}\delta^2(A)\Pi_{i\in A}\hat{p}_i(1-\hat{p}_i)}} = t(P)$$

where:

$$\hat{h}(P) = \frac{\sum_{t=1}^{m}\phi[X(w_t)]}{m}$$

$$\hat{p}_i = \frac{1}{m}\sum_{i=1}^{m} X_i$$

Proof:

$\phi(w_t)$ is a Bernoulli random variable with weight $w_i = \frac{1}{m}$. If $P = (p_1,...,p_n)$ is unchanged during the experiment then $h(P) = E_\mu[\hat{h}(P)]$. Take the weights

$$\theta_i = \frac{w\, w_i}{\sum_{i=1}^{n} w_i^2}$$

where $w = \sum_{i=1}^{n} w_i$ then m is also the 'equivalent sample size'. Then t(P), for a sufficiently large m follows a T-Student distribution with approximatelly m - 1 degrees of freedom, see Bouza (1995) an the proposed test can be performed♣

3. SHOCK MODELS

A system can be affected in its performance by external causes that we will denominate as 'shocks'. Formally it is given by:

Definition 2. A shock is an application $S_A^j \mapsto v(A)$ that represents an exogeneous cause that destroys all $i \in A$.

Note that the influence of internal causes is not modeled. For example this, means that the products are out of the 'infant mortality period' or that the coalition does not reach to a consensus because of discrepancies among the coalitioned agents. Following the ideas of Gasemyr-Natvig (1995) we have:

$\aleph_A = \left\{S_A^j : j = 1,...,J\right\}$, set of shocks for A

E_i = {B: i is shocked}, set of shocks for i.

$E_A = \{B : B \cap A \neq \Phi\}$

The status of A is evaluated by the Bernoulli random variable:

$$Y_A = \begin{cases} 1 & \text{if} \quad S_A \in E_A \\ 0 & \text{otherwise} \end{cases}$$

with $E(Y_A) = \theta_A$.

Currently the shocks can be considered as mutually independent. Real life problems behavior are modeled by the observation of a sequence a of signals. The structure of X_i leads to considering it as a function of the Y_A's. They are consistent with X_i. The discrete product measure prior

$$p_i = P(X_i : w) = \Pi_{A \in E_i} Q(Y_A) = \Pi_{A \in E_i} q_A$$

can be used for Bayesian inferences. q_A is an unspecified marginal for Y_i. This is a Bayesian approach that has been used by several authors in Sampling theory, see Chadhuri-Voos (1988).

The sequence of signals are characterized by $Y \in \{0,1\}^{|a|}$. If the DM assigns a probability $q \in [0,1]^{|a|}$ we can model the functioning of it by

$$X_i = \Pi_{A \in E_i} Y_A$$

The system can be redefined by using the function

$$\varsigma : \{0,1\}^{|a|} \rightarrow \{0,1\}^n$$

such that $\varsigma(Y) = X$,

$$\varsigma * : \{0,1\}^{|a|} \rightarrow [0,1]^n$$

and $\varsigma*(Y) = q$.

Take $A* = \cup_{j=1}^{n} \{j\}$. An structure $\varphi : \{0,1\}^{|a|} \rightarrow \{0,1\}$. Then $\varphi(Y) = \phi \circ \varsigma(Y)$ permits to identify the components of the system. We can express $\varphi(Y) = \phi(\varsigma(Y)) = \Sigma_{A \in \aleph} \Pi_{i \in A}(\Pi_{B \in E_i} Y_B) = \Sigma_{A \in \aleph} \delta(A) \Pi_{B \in A} Y_B$.

We avoid the use of natural numbers for characterizing the initial system by using $\Theta = \{\aleph^*, \varphi\}$. The components of Θ are independent because φ is Y - evaluated. The realibility of Θ is

$$g(\theta) = \Sigma_{A \in \aleph} \delta(A) \Pi_{B \in E_A} \theta_B$$

Hence the procedure used previously permits to predict in this framework under the hypothesis of iid because

$$E_\mu[\varphi(Y)] = g(\theta)$$

Note that the knowledge of the signed domination function $\delta(A)$ for (ζ,ϕ) permits to predict the realibility of the dependent case.

Take the set of shocks for the subsets a of ζ and a family $\aleph$ of them. Then:

$$\hat{g}(\theta) = \varphi(Y)$$

is model unbiased and its error

$$V_\mu[\hat{g}(\theta)] = \Sigma_{A \in \aleph} \delta^2(A) \Pi_{B \in E_A} \theta_A (1 - \theta_A)$$

is estimated by

$V_{\mu}[\hat{g}(\theta)] = \Sigma_{A\in\aleph}\delta^2(A)\Pi_{B\in E_A}\hat{\theta}_A(1-\hat{\theta}_A)$

where

$$\hat{\theta}_A = \frac{1}{m}\sum_{i=1}^{m} Y_A^i$$

If a random experiment is performed and the results

$Y_A^t(w_t), t = 1,...,m$ are observed then $\left]\hat{g}(\theta) - t\left(m-1, 1-\frac{\alpha}{2}\right)\sqrt{\hat{V}\mu[\hat{g}(\theta)}, \hat{g}(\theta) + t\left(m-1, 1-\frac{\alpha}{2}\right)\sqrt{\hat{V}\mu[\hat{g}(\theta)]}\right[$ is a confidence interval at the confidence level α.

4. EXAMPLES

The described systems can be used for modeling different practical problems. We describe some of them.

Example 3. *Evaluation of a email network.* Suppose that we have k different boxes. The network consists of n devises for sending a message. We can identity $\aleph = U_{j=1}^{k}A_j$ with the network. A_j is a path, p_i is the probability that a component i works. $\delta(A_j)$ describes the efficiency of the path. Once the message is successfully sent $\phi(X) = 1$. Though p_i hould be given by the producer of the devises the real problem may be such that the operational conditions are far from being accomplished. The independence of the ' boxes' in an email network can be accepted. The DM evaluates the technical condition of the devises and establishes the values of $X_1,...,X_n$. The h(p) can be predicted and the comunication network is evaluated. The DM uses the evaluation for establishing the realibility of sending successfully a message. The existence of shocks is present when damages in the lines or the computers are expected or changes in the electric power are usual.

Example 4. *Design and operation of a monitoring network.* The DM studies the environment of a region. A network of monitoring stations is designed. Each station i obtains information from a set of contaminating sources $C^i = \left\{c_1^{(i)},...,c_H^{(i)}\right\}$ and provides information on a set of levels of a pollutant $\Lambda = \left\{L_1^{(i)},...,L_I^{(i)}\right\}$. The conditional probabilities $\text{Prob}\left(c_j^{(i)} : L_t^{(i)}\right)$ characterizes the relation between the observed level of the pollutant and the sources of it. For a certain message X the DM computes

$\phi(X) = \Sigma_{A\in\aleph}\delta(A)\Pi_{i\in_A}X_i$

where $\phi(X) = 1$ establishes that the environment is being seriously affected. A is related with a monitoring station and

$$X_i = \begin{cases} 1 & \text{if} \quad \text{the i observed } L_h^i \text{ is 'large'} \\ 0 & \quad \textit{otherwise} \end{cases}.$$

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